c. What will you read in the following paragraphs?

The paragraphs 2 through 4 describe the beginning of the spiral on derivatives. In paragraph 2, the concept of *derivative in one point* is introduced from several contexts. Whether it concerns the local slope of a mountain, instantaneous velocity, the slope of a tangent line, or a marginal cost, mathematically speaking, the same idea lies behind it: the concept of derivative. In paragraph 3, the derivative is examined at an "arbitrary" point, which yields a *derivative function*. In paragraph 4, derivatives of power functions and polynomial functions are *calculated*. The remainder of the spiral is not elaborated in detail: deriving new functions with new calculation rules, applications related to the evolution of various functions, extremum problems, etc.

The following paragraphs discuss a second spiral that does not chronologically follow the one on derivatives but is intertwined with it: the spiral of limits. If we abandon a preliminary chapter on "training in limit calculations", how do we apply limits to the different classes of functions? In paragraph 5, we zoom in (or rather, out) on limits related to the "behavior at infinity" of functions and sequences. Finally, paragraph 6 discusses the famous "indeterminacy".

2. Giving sense to "derivative"

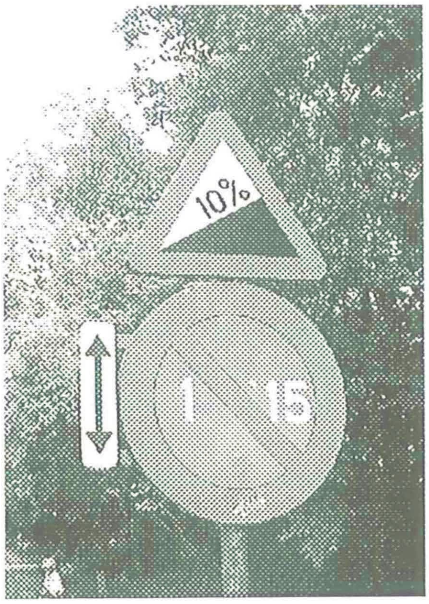
a. Four examples

The concept of "derivative" is a rich one. To ensure that students also grasp this concept, we must introduce them to several situations typical of it. These situations should serve as a kind of model for the concept of "derivative" and come from both within and outside mathematics. Because "derivative" is such an important concept, the time required for this is certainly time well invested.

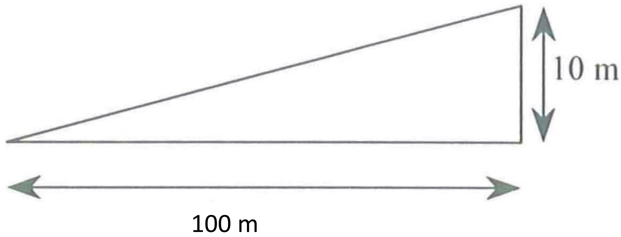
First example: the slope of a mountain

For the first example, we will use what the students in the second grade saw about the slope of a line.

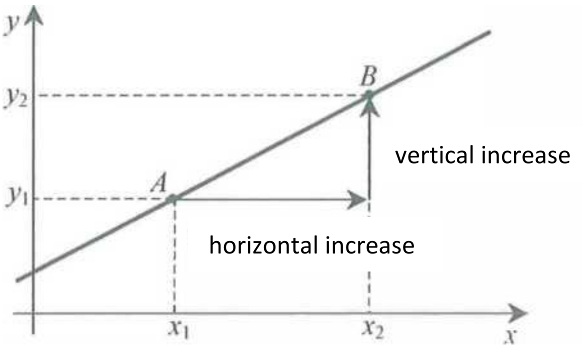
The slope of a mountain

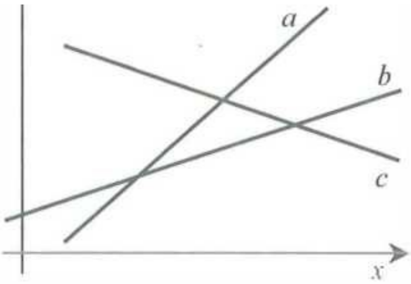


At the top of the photo, you see a traffic sign warning road users that a steep slope awaits them. You probably know the meaning of this 10%. It means that for a horizontal increase of 100 m, there's a vertical increase of 10 m. Schematically:



The ratio $\frac{10}{100} = \frac{\text{vertical increase}}{\text{horizontal increase}}$ is a number that measures the slope. You already encountered this in the second grade. You also learned how to measure the slope of a line in a coordinate system. This means you can express the strength of the slope with a number. This number is called the *slope* or the *steepness* of the line.

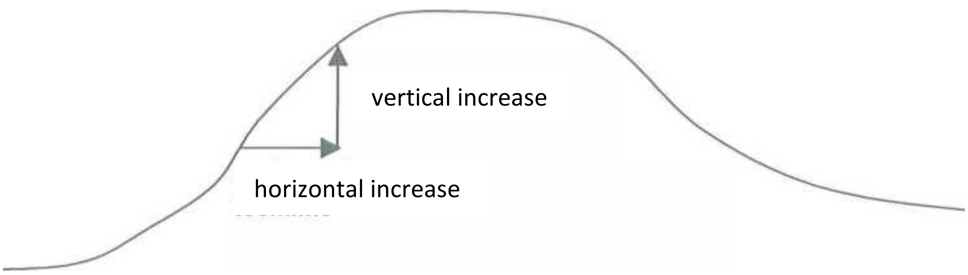




1. Do you still remember the formula for finding the slope of the line through the points $A(x_1, y_1)$ and $B(x_2, y_2)$?

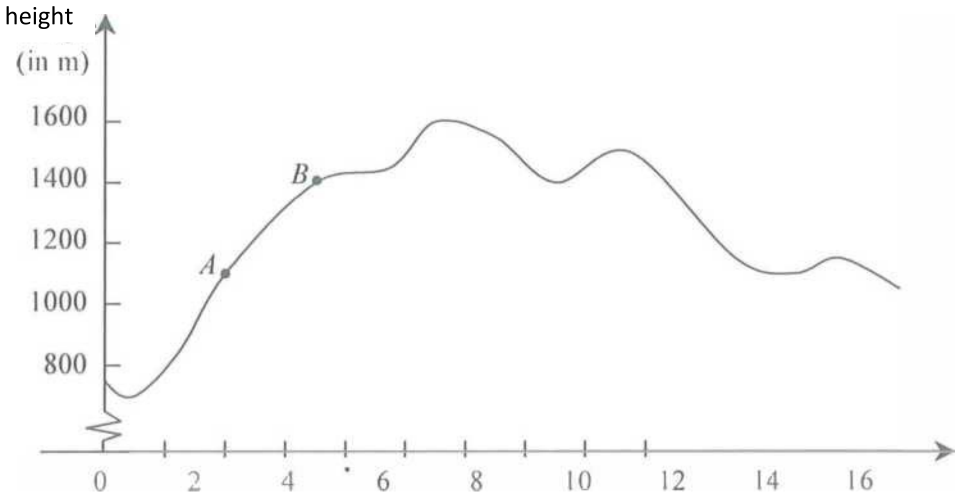
2. In the figure on the right, three lines are drawn.
   a. Line *a* is steeper than line *b*. What does this mean for their slopes?
   b. Line *a* rises, while *c* drops. How can you tell this from the slopes?

We return to the slope of a road. If the mountain slope isn't straight, we can no longer speak of the slope. We then speak of the slope at a point. We can also speak of the average slope. To calculate the average slope between two points, you pretend the two points are connected by a straight line and you calculate the slope of that connecting line.



The ratio $\frac{\text{vertical increase}}{\text{horizontal increase}}$ provides a measure of the average slope.

The slope can vary considerably within the section under consideration. Below, you see a graph representing a road on a mountain slope. The X-axis represents the horizontal distance traveled, and the Y-axis represents the elevation at which you are now located.



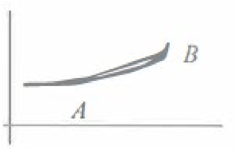
3. Calculate the average slope between point *A* and point *B*.
4. Near point *A*, the slope is very steep. How can we get a better idea of the slope at that point? (*Taking a smaller interval*)

A skier experiences first-hand what the slope is like *at a certain point*.

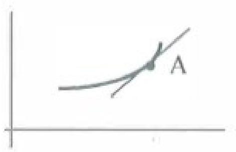
5. Draw on the graph as accurately as possible the (direction of the) skis of a skier at point *A*.
6. Calculate the slope of the skis.
7. Review the formula from exercise 3 you used tc calculate the average slope. You'll recognize the slope coefficient formula in this formula. From which line?
8. Did you also calculate the slope of a line in exercise 6? Which one? What would you call that line? *(We calculated the slope coefficient of a line that follows the slope of the mountain as closely as possible at point A. This line is called the tangent line.)*

We remember:

- the average slope between A and B is the slope of the line joining points A and B; this is the slope of the line AB;



- The local slope (or slope at a point) is the slope of the tangent line; to calculate it, first draw the tangent line on the figure.



Second example: tangent line to a circle

This brings us to the concept of "tangent line". Students are already familiar with tangent lines to a circle from their second-grade geometry lessons. The following workbook bridges the gap between the tangent line to a circle and the tangent line as it appeared in the workbook above.

Tangent line to a circle

In the second grade, you studied tangent lines to a circle. You learned, among other things, that the tangent line at a point on a circle is always perpendicular to the radius through that point. This allows us to always find the equation of the tangent line at a given circle and point.

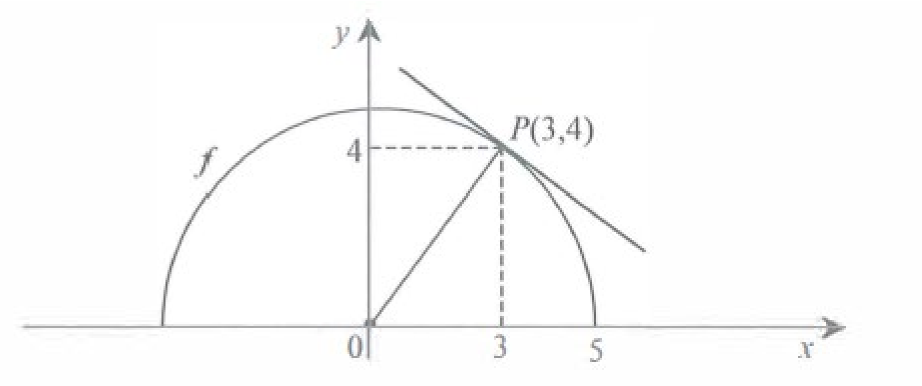
Now consider the circle with equation $x^2 + y^2 = 25$. The point $P(3, 4)$ lies on this circle.

1. Determine the equation of the tangent line at P to the circle. What is the slope of the tangent line?

   (Slope tangent line = $-\frac{3}{4}$*)*

Note that the tangent line in $P(3, 4)$ is the only line through P that intersects the circle in only one point. All other lines through P intersect the circle in two points.

Now let's look at it in another way. The part of the circle that lies above the x-axis is the graph of the function $f(x) = \sqrt{25 - x^2}$.

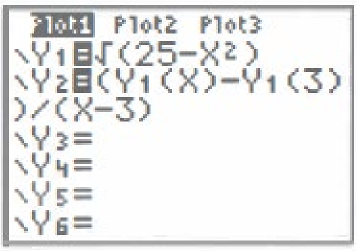


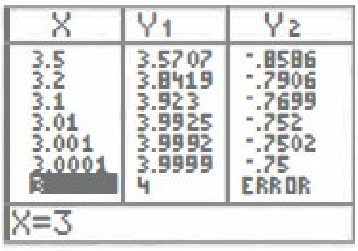
2. Calculate the slope of some lines connecting *P* to another point on the graph. Create a table using your graphing calculator and complete the tables below.

x	$f(x)$	slope of the line through (3, 4) and (*x, f(x)*)
3,5		
3,2		
3,1		
3,01		
3,001		
3,0001		

x	$f(x)$	slope of the line through (3, 4) and (*x, f(x)*)
2,5		
2,7		
2,8		
2,9		
2,99		
2,999		

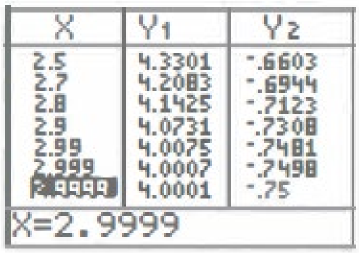
(In response, students will see screens similar to the following. By selecting Independent ASK in TableSet, you can ensure that you can enter the desired x-values yourself.)





X	Y1	Y2
3.5	3.5707	-.8586
3.2	3.8419	-.7906
3.1	3.923	-.7699
3.01	3.9925	-.752
3.001	3.9992	-.7502
3.0001	3.9999	-.75
3	4	ERROR

X=3



X	Y1	Y2
2.5	4.3301	-.6603
2.7	4.2083	-.6944
2.8	4.1425	-.7123
2.9	4.0731	-.7308
2.99	4.0075	-.7481
2.999	4.0007	-.7498
2.9999	4.0001	-.75

X=2.9999

3. What do you notice?

(The slope values of the connecting lines get closer to the slope of the tangent line as x approaches 3.)

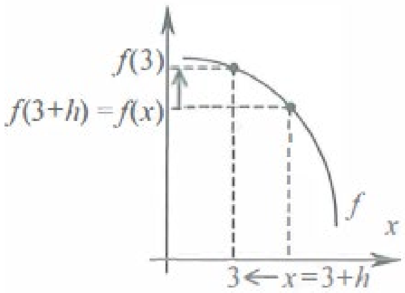
In mathematics we say that $-\frac{3}{4}$ is the *limiting value* of $\frac{f(x)-f(3)}{x-3}$ as *x* approaches 3. The usual notation for this is

$$\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}=-\frac{3}{4}. \qquad (1)$$

The fraction $\frac{f(x)-f(3)}{x-3}$ is the slope of the connecting line of (3, 4) and (*x, f(x)*), or the slope of the graph between 3 and *x*. By

taking *x* closer to 3, this slope increasingly approaches the *slope* or the *slope* of the *tangent line* at point (3, 4), which we had already calculated based on geometric properties. The limit calculation therefore yields the same result as the previous, geometric method.

4. The slope of the tangent line is negative. Can you relate this to the graph's slope?
   (The tangent line is decreasing, so the graph is decreasing near (3, 4).)



We've taken values of *x* that don't deviate too much from 3.
By changing the notation slightly, we can express this more clearly.
We'll write the *x* from the limit expression above as 3 + *h* from now on.
Here, *h* represents any number other than 0. However, we're primarily interested in the expression where *h* is very small, namely, $h \approx 0$.

5. Replace *x* with 3 + *h* in the expression $\frac{f(x)-f(3)}{x-3}$.

   (This gives $\frac{f(3+h)-f(3)}{h}$*)*

6. What happens to *h* when *x* ->3?

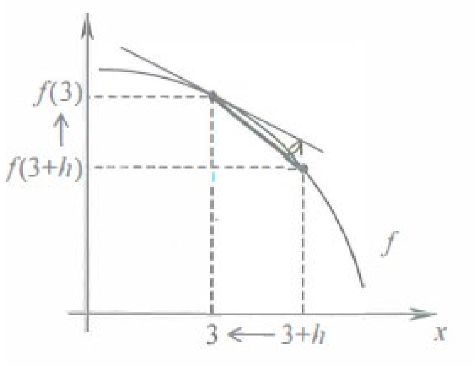
Formula (1) then becomes

$$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=-\frac{3}{4}. \qquad (2)$$

The limit method also allows us to determine the slope of the tangent line for other functions (where we do not have a geometric method).

7. Using a table, determine the slope of the graph of $f(x)=2^x$ at point (2, 4). Record your findings with an expression of the form (1) or (2).

We remember:



We can calculate the slope at a point on the graph of a function using a limit procedure. To find the slope of *f* in, for example, (3, *f*(3)), calculate the average slope of the graph between the points (3, *f*(3)) and (3+*h*, *f*(3+*h*)) for values of h that are increasingly close to 0. You allow 3+*h* to approach 3, so to speak. This "approaching" must occur both to the left and to the right of 3. The value the slope tends to reach is called the *limit value*.
We denote this by

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

This limit value is *the slope of the graph at that point* or the *slope of the tangent line* at that point.

From now on, we will always work with the second notation for the limits (the one with *h* -> 0). In the first two contexts, it was important to connect with what the students know. In expression (1), you recognize the structure of a slope coefficient better than in expression (2). But if you have to do calculations with these expressions, the second form offers more advantages. For example, when calculating derivative functions, there is no doubt about the meaning of *x*.

Third example: velocity

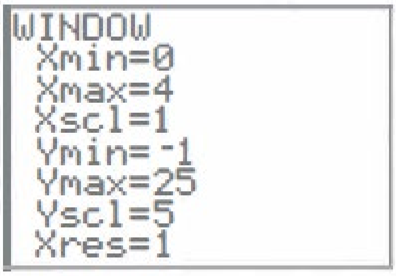
A crucial context when introducing derivatives is *velocity*. Velocity is essentially a *derivative with respect to time* and can take various forms: the concept from physics, population growth rate, capital growth rate, etc. In the following workbook, we have chosen the physical velocity.

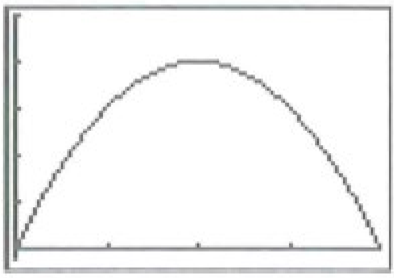
Velocity

A stone is thrown vertically upward. The height of the stone is given by:

$$f(t) = -5t^2 + 20t$$

where *t* is the time in seconds from the start of the throw, and *f(t)* is the height in meters at instant *t*. We draw the graph of *f*.





We see that the height of the stone first increases and then decreases again. We now want to know the stone's *velocity* one second after the throw. First, we calculate the stone's average speed during a time interval near t = 1.

1. Calculate the height difference the stone travels from t = 1 to t = 2.
2. Calculate the average speed in the time interval [1,2].
3. Does this average velocity say much about the velocity at time t = 1?
   (No, in the interval [1,2]*, the increase in height is greater at the beginning than at the end. The average velocity in that interval is influenced too much by what happens at the end of this time interval.)*
4. How can you get increasingly better approximations for the velocity at t = 1?
   (By considering increasingly smaller time intervals.)

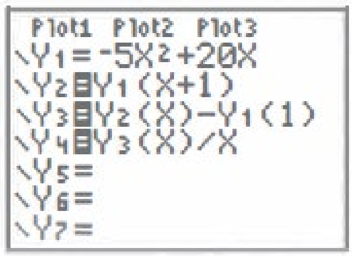
We calculate the average speed of the stone between t = 1 and an instant that differs only slightly from this, namely t = 1 + h.

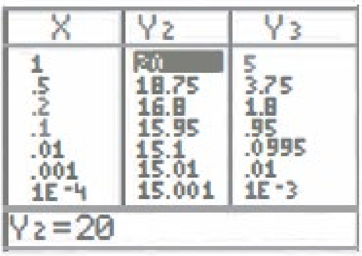
5. Calculate the average speed for each of the time intervals below. Use your graphing calculator to create a table like the one below.

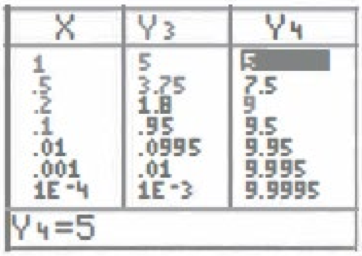
time interval	h	$f(1+h)$	height difference $=f(1+h)-f(1)$	average velocity $=\dfrac{f(1+h)-f(1)}{h}$
[1, 2]	1	20	5	5
[1; 1,5]	0,5			
[1; 1,2]	0,2			
[1; 1,1]	0,1			
[1; 1,01]	0,01			
[1; 1,001]	0,001			

time interval	h	$f(1+h)$	height difference $= f(1+h) - f(1)$	average velocity $= \dfrac{f(1+h)-f(1)}{h}$
[0,1]	–1			
[0,5;1]	–0,5			
[0,8;1]	–0,2			
[0,9; 1]	–0,1			
[0,99; 1]	–0,01			
[0,999; 1]	–0,001			

(The table must be constructed carefully. The TI-83 only recognizes one independent variable for functions, namely X. Both the variables t and h must therefore be represented by X. It is best to work in steps as shown in the screens below.)







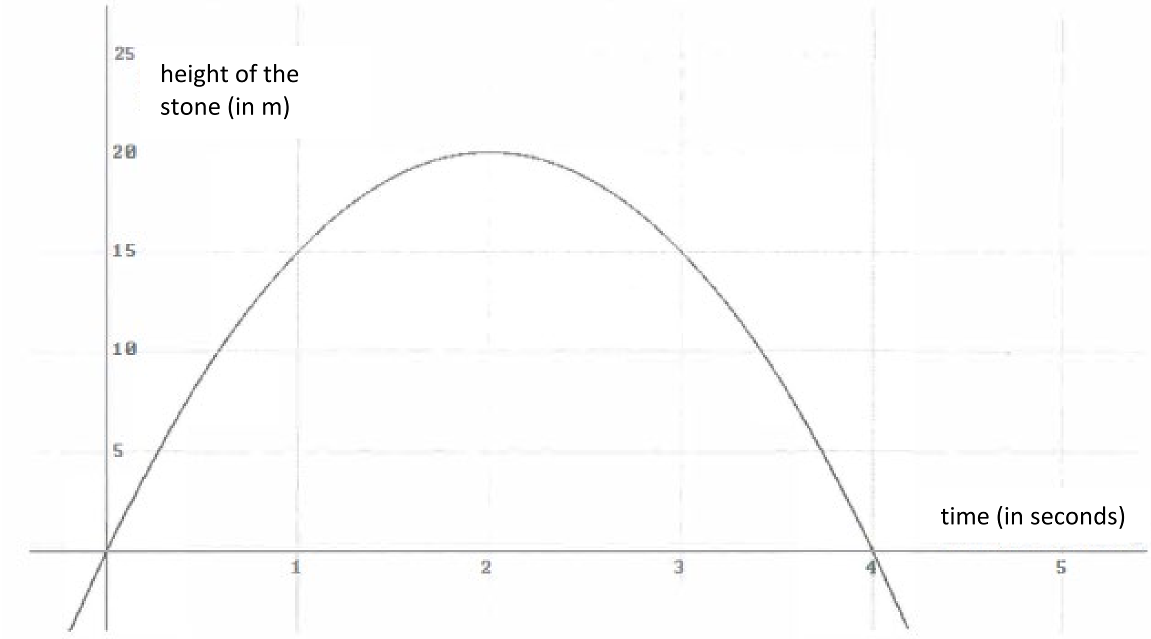
If we give h an increasingly smaller positive value (≠0) (or a negative value with an increasingly smaller absolute value), the average velocity in the time interval [1, 1+h] (or [1+h, 1] if h is negative) $\dfrac{f(1+h)-f(1)}{h}$ gives an increasingly better approximation of the instantaneous velocity of the stone at time 1 second after the throw. Based on the table, we suspect that this instantaneous velocity is 10 m/s.

We say that 10 is the limiting value of the average velocity $\dfrac{f(1+h)-f(1)}{h}$ as h approaches 0. We denote this as

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = 10 .$$

We now look for a meaning for the average and instantaneous velocity on the graph.

6. Can you also view the average velocity $\frac{f(1+h)-f(1)}{h}$ as the slope of a straight line? If so, which straight line?



7. Of which straight line is the instantaneous velocity the slope? Draw this straight line on the graph above.

8. Which straight line can you use to illustrate the instantaneous velocity after 1.5 seconds? Draw that straight line and compare it with the straight line from exercise 7. When is the instantaneous velocity greatest: at t = 1 or at t = 1.5? Can you also see this without drawing the tangent lines?

9. Now explain using the graph: the stone's speed first decreases, at the highest point its speed is 0, and as it falls, its speed gradually increases.

We remember:

- the instantaneous velocity is the limit of the average velocity at increasingly shorter time intervals;
- the average velocity over a certain time interval is the slope of the line connecting the starting and ending points;
- the instantaneous velocity is the slope of the tangent line.

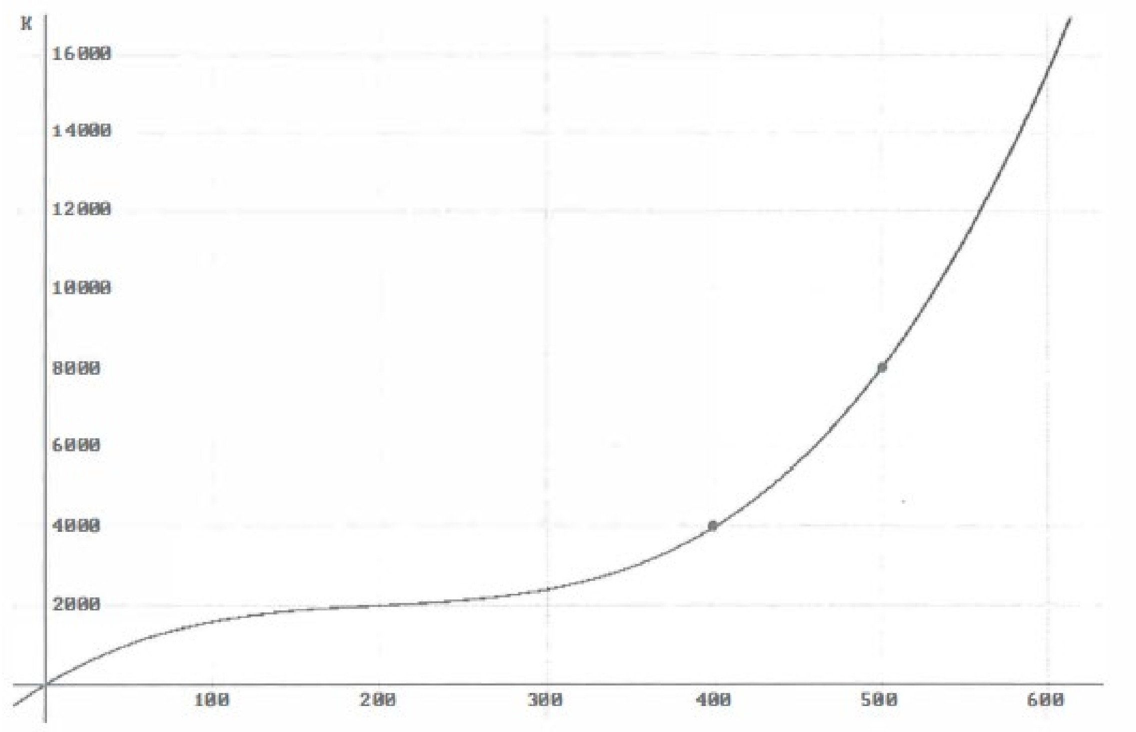
Fourth example: marginal cost

Velocity isn't the only application of derivatives. We illustrate this with an economic application. In the workbook, we assume that students are not yet familiar with the concept of "marginal cost" from their economics classes. In economics courses, this concept already has a discrete meaning.

In that case, it makes more sense to make the connection with the continuous meaning later. In the workbook below, we reason on the graph without the corresponding formula. Unlike the situation with (instantaneous) speed, students do not have an intuitive grasp of the concept of "marginal cost". We now use the process of approximating instantaneous change by averaging changes over increasingly smaller intervals *to introduce a new concept*.

Marginal cost

A sugarcane producer markets his product for 56 francs per kg. The profit he makes depends on the number of kg he can sell, but also on the costs incurred to produce this quantity (purchase, manufacturing, storage). Consequently, the total costs depend on the quantity to be produced. The quantity of sugarcane to be produced is denoted by *q* and is expressed in kg. The costs (expressed in francs) for a production quantity *q* are denoted by *K(q)*. The producer is interested in how costs change as production increases. The figure below shows the graph of the total cost function *K*.



1. Calculate the cost increase (or total cost increase) when production increases from 400 to 500 kg. This cost increase is expressed in francs
(4,000 francs).

2. The average cost increase for a production increase from 400 to 500 kg is the average additional cost per kg to increase production from 400 to 500 kg. Calculate the average cost increase for a production increase from 400 to 500 kg. What unit is the average cost increase expressed in?
   (40 francs/kg)

3. Can you interpret this average cost increase on the graph?

   (The average cost increase is the slope of the line connecting (400, 4,000) and (500, 8,000).)

In economics, the slope of the cost graph at a point is also significant. This slope indicates how much and in which direction costs change as production changes starting from a production level of 400 kg. The slope at a point on the cost graph is called *the marginal cost*. Literally, this means the cost added at the edge (or more accurately, "in the margin"). On the graph, the marginal cost, like the instantaneous rate in the previous example, is the slope of the tangent line.

4. Using the graph, estimate the marginal cost at a production q = 400.

   (Draw at sight the tangent line to the graph at (400, 4,000) and calculate the slope of this line.)

We remember:

- the marginal cost is the limit of the average cost increase for ever smaller production increases;
- the average cost increase over a given production increase is the slope of the line connecting the starting and ending points;
- the marginal cost is the slope of the tangent line.

b. One concept

The four examples we worked out above serve as model situations for the concept of "derivative". In the following workbook, we will again emphasize the analogy and introduce the new concept.

The derivative

In the examples above, we encountered four situations that bear some resemblance. Each involves a measure of change. The diagram below summarizes the examples.

function	average change	taking a limit	instantaneous change
① height of the road as a function of the horizontal distance covered	average slope of the road = slope of the intersection line	-------------→	local slope (ski slope) = slope of the tangent line
② function (semicircle)	slope of the connecting line	-------------→	slope of the tangent line
③ height of a stone	average velocity = slope of the cutting line	-------------→	instantaneous velocity = slope of the tangent line
④total cost function	average cost increase = slope of the cutting line	-------------→	marginal cost = slope of the tangent line
function	average change = slope of the intersection line	-------------→	instantaneous change = slope of the tangent line

These four examples share a striking similarity: there's a mathematical concept behind them. We can view the average change in each case as the slope of a line of intersection, and the instantaneous change on the graph can be seen as the slope of the tangent line. In mathematics, this slope of the tangent line is called the *derivative of the function at a point*.

Definition

The *average change* of a function *f* is given by the (difference) quotient

$$\frac{f(a+h)-f(a)}{h}.$$

This is an approximation of the *instantaneous change*. This approximation consistently improves when $h \to 0$.

The *derivative* of a function *f* at a point *a*, denoted *f'(a)*, is a measure of that instantaneous change. In short:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

The difference quotient $\frac{f(a+h)-f(a)}{h}$ is often abbreviated as $\frac{\Delta y}{\Delta x}$.

'Δy' is read as "the difference of the *y*-values" and 'Δx' as "the difference of the *x*-values."

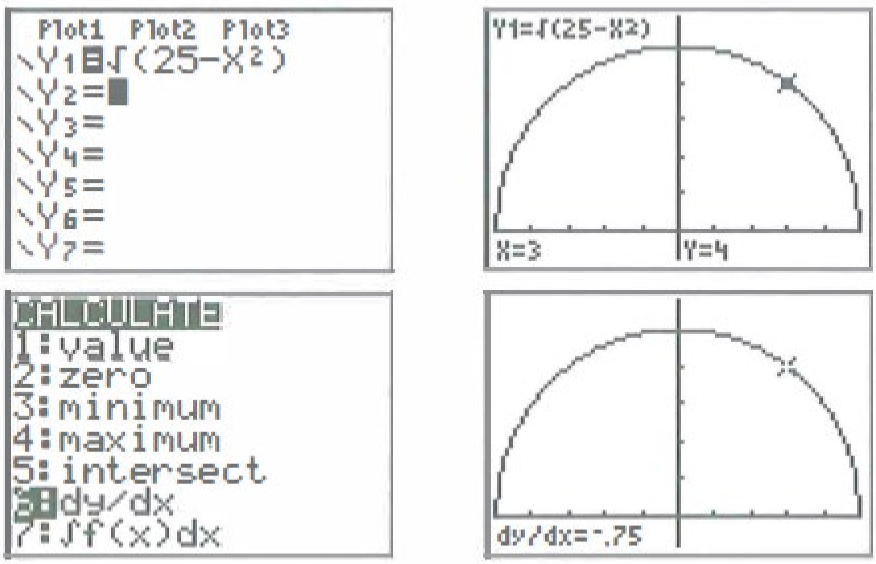
There's another notation for *f'(x)* that's related to this, namely $\frac{dy}{dx}$.

This notation offers the advantage of allowing you to specify which variable to differentiate. For the marginal cost, for example, we get $\frac{dK}{dq}$. This notation is also used on most (graphing) calculators that have the "differentiate" function. This is the case, for example, with the Tl-83. Using the function from the second example, we'll show you how to calculate the derivative at a point on that calculator.

function: $f(x) = \sqrt{25 - x^2}$

requested: *f'(3)*

working method: Enter the formula in the Y= menu and draw the graph.
Now, from the 2nd CALC menu, select 6:dy/dx. Mow you can move the cursor over the graph until you reach the desired *x*-value, or simply enter the correct *x*-value (here: *x* = 3). Press Enter, and you'll get the value of the derivative in 3.



1. Now use your calculator to check the velocity (after 1 second) of the stone from the third example.

2. Use your calculator to calculate the velocity at which the stone hits the ground.

3. Calculate the stone's speed after 2 seconds. Can you explain this?
(The speed is 0. The tangent line at (2, 20) is horizontal.)

4. Given is the distance traveled as a function of the time for a car ride:

$$s(t) = 150t^2 - 50t^3 \qquad t \in [0, 2]$$

(t in hours and s in km).

a. Calculate the average speed over the entire trip.
b. Calculate the average speed during the first fifteen minutes of the trip.
c. Use your graphing calculator to calculate the speed at time $t = 0.25$.
The first half hour of the trip passes through a built-up area. Is the driver in violation?

5. The sugarcane manufacturer created a mathematical model of his total cost function. He found

$$K(q) = 0{,}0002q^3 - 0{,}12q^2 + 26q \qquad q \in [0, 600]$$

a. Use your graphing calculator to calculate the marginal cost at a production $q = 400$.
The cane sugar is sold at 56 francs per kg. For the following questions, you can assume that everything produced is also sold.
b. Calculate the profit $W(q)$ in function of the produced quantity q.
c. Calculate the average profit increase when production increases from 400 to 500 kg.
d. What is meant by *marginal profit*? Calculate that marginal profit at a production of 400 kg.

Of course, we don't have to limit ourselves to polynomial functions for these exercises. However, students can't yet solve questions like "At what production is profit maximized?" algebraically. For that, you need the derivative function. The following problem also can't be fully solved yet.

Tangent line parallel to a given line

Given the function $f(x) = x^3$. At what point is the tangent line parallel to the line $y = 2x$?

Hint: Graph f and experiment both on paper and with your calculator.

With their current knowledge, students can only solve this problem (very) imprecisely by shifting their ruler parallel until they find the approximate position of the tangent. Using their calculators, they arrive at a solution close to 0.82. They are unlikely to find the exact solution this way.