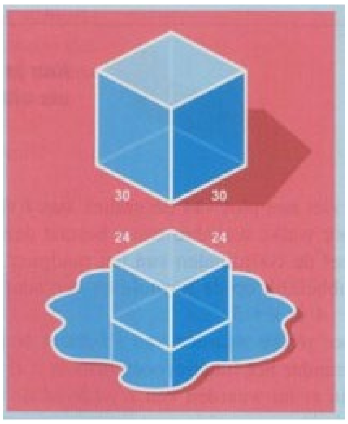
Exercise A: The ice cube

An ice cube with edges of 30 mm long starts to melt down slowly. Every minute, the edges get 1.5 mm shorter. The volume of the ice cube is described by the formula
$V = (30 - 1{,}5\ t)^3$, where V stands for the volume in mm^3 and t for the time in minutes.

a. Calculate the volume of the ice cube when t=0.
b. What are meaningful values for t? And for V?
c. Plot and sketch that part of the graph for which the variables are meaningful.
d. Trace the graph with the cursor and investigate after how many minutes the volume is less than 10 000 mm^3. Provide your answer with a precision of one decimal.



Given:

- An ice cube with edges 30mm long;
- Every minute the ice cube melts down and the edges become $1{,}5mm$ shorter.
- The volume V of the icecube in mm^3 as a function of the time t in minutes is $V(t) = (30 - 1{,}5t)^3$

Solutions

a. $V(0) = (30 - 1{,}5 \cdot 0)^3 = 30^3 = 27\ 000$

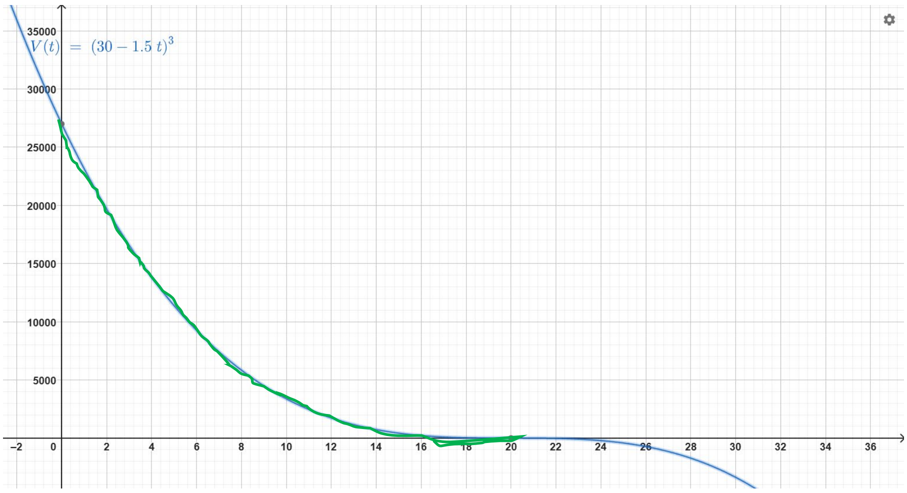
Answer: The volume of the ice cube in the beginning when $t = 0$ is 27000 mm^3 or $27cm^3$

b. The time nor the volume can be negative, therefore $t \geq 0$ en $V \geq 0$
We calculate when $V = 0$

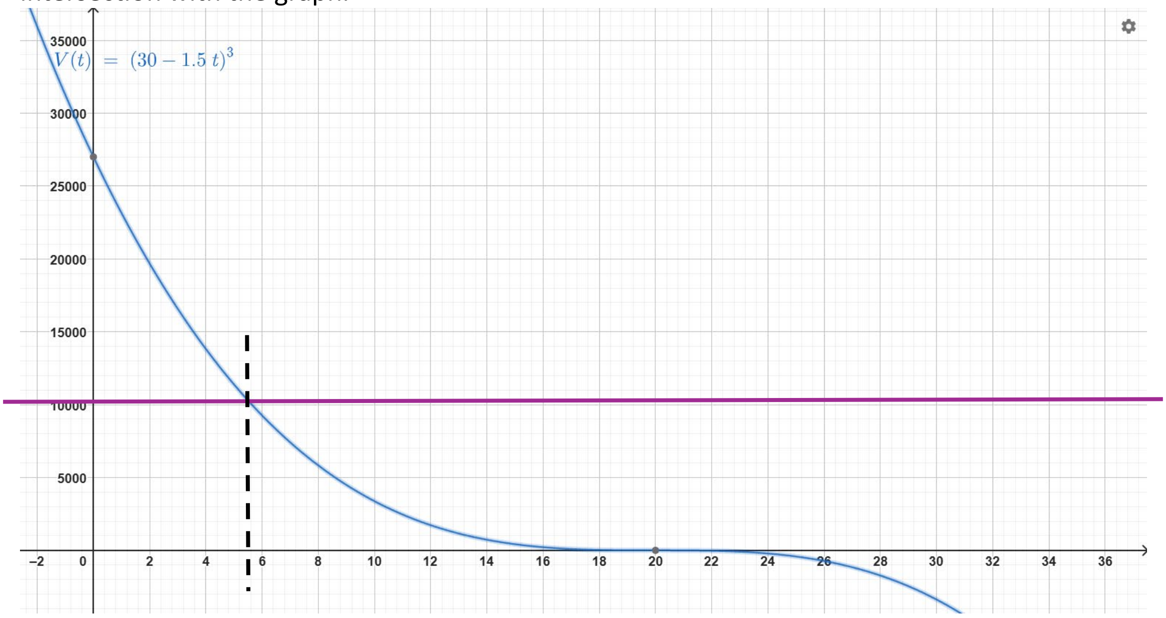
$$V(t) = 0 \Leftrightarrow (30 - 1{,}5t)^3 = 0 \Leftrightarrow 30 = 1{,}5t \Leftrightarrow t = 20.$$

Answer: $t \in [0{,}20]$ and $V \in [0{,}27000]$

c. Answer:
The part of the graph for which the values of t and V are meaningfull, is indicated on the graph in green:



d. We solve this question both by using the graph and algebra.
Using the graph (not exact): We draw the straight line $y = 10000$ and estimate the x-value of its intersection with the graph.



We estimate that from $t = 5{,}3$ thus after 5,3 minutes, the volume will be smaller than $10000mm^3$

Algebraic: $V(t) < 10000 \Leftrightarrow (30 - 1{,}5t)^3 < 10000$

$$\Leftrightarrow 30 - 1{,}5t < 10\sqrt[3]{10} \Leftrightarrow 30 - 10\sqrt[3]{10} < 1{,}5t \Leftrightarrow 8{,}46 < 1{,}5t$$
$$\Leftrightarrow 5{,}64 < t.$$

We must round up the number 5,64 since the volume must be smaller than 10000. If we round down to $t = 5{,}6$ minutes, the volume will not be smaller than $10000mm^3$

Answer: The volume of the ice cube will be smaller than $10000mm^3$ after 5,7 minutes.

Exercise B: The temperature in a cool storage

The temperature in a cool storage room is given by the following function:

$$T(t) = \frac{3t^2-6t+3}{t^2-2t+2}$$

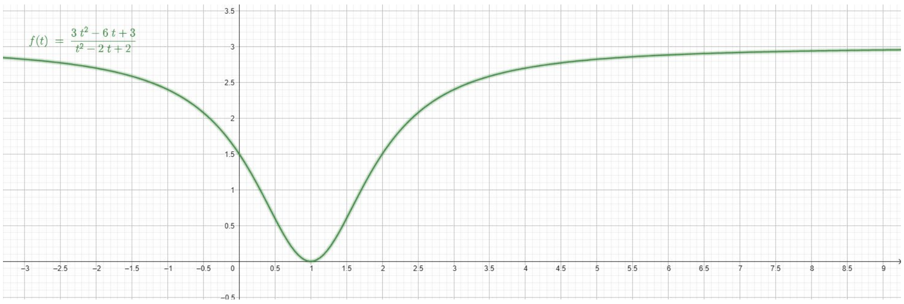
T = temperature (°C); t = time (in hours); t = 0 corresponds to 3 a.m.

a. Sketch the graph of this function
b. If the temperature drops below 1 °C, there is a risk of damage to the food. How long was the temperature below 1 °C? From when to when?
c. When did the temperature start to rise again?
d. What temperature does the cool storage evolve to?

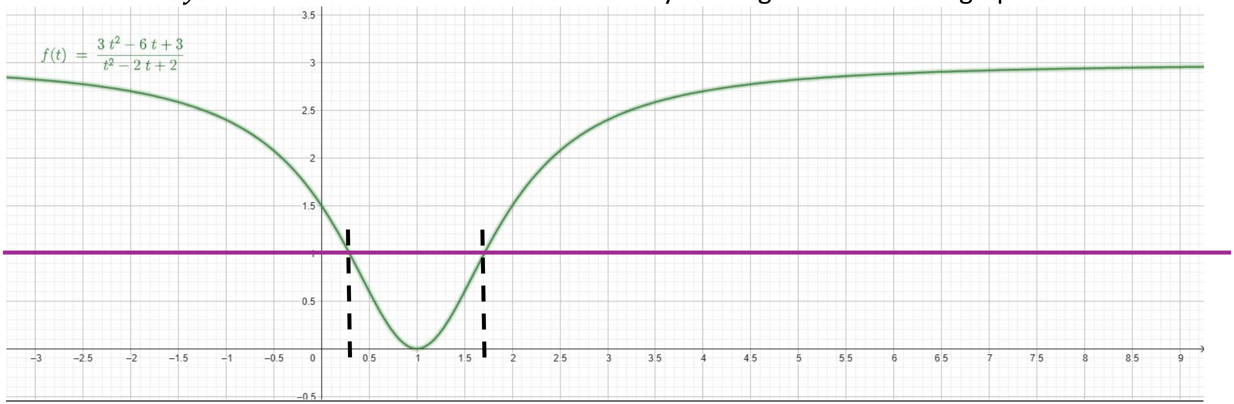
Given:

- The temperature (°C) as a function of time (hours) is $T(t) = \frac{3t^2-6t+3}{t^2-2t+2}$
- We start measuring time at 3 a.m., so than $t = 0$. We don't know the date, only the time in a day.

a. The graph:



b. On the graph we can estimate how long the temperature drops below 1°C. We first draw a horizontal line $y = 1$. Then we estimate the x-values by reading them form the graph.



We read that this is approximately between 0,3 hours and 1,7 hours. $0{,}3\text{h} = 0{,}3 \cdot 60 = 12 \text{ min}$ and $1{,}7 \text{ h} = 1{,}7 \cdot 60 = 104 \text{ min} = 1{:}44'$. Since $t = 0$ corresponds to 3 a.m. the temperature in the cool storage is between 3:12' and 4:44' below 1°C.

We calculate this algebraic

$$T(t) = \frac{3t^2-6t+3}{t^2-2t+2} < 1 \Leftrightarrow 3t^2 - 6t + 3 < t^2 - 2t + 2 \Leftrightarrow 2t^2 - 4t + 1 < 0$$

Let's calculate where $t = 0$. We use the method of the discriminant. $D = 16 - 8 = 8$

$$t_{1,2} = \frac{4 \pm \sqrt{8}}{4} = 1 \pm \frac{\sqrt{2}}{2} = 1{,}71 \quad \text{of } 0{,}29$$

On the graph we notice that in between these values for t the graph goes under the horizontal line $y = 1$. For the algebraic calculation, we need to check the sign table and indeed since $a > 0$, the sign in between the zero points will be negative ($2t^2 - 4t + 1 < 0$), meaning $T(t) < 1$.

For question 1: $1{,}71 - 0{,}29 = 1{,}42 = 1h + 0{,}42 \cdot 60' = 1h25'$

For question 2: $1{,}71 = 1 + 0{,}71 \cdot 60 = 1h43'$ and $0{,}29 \cdot 60 = 17{,}4 = 17'$

Answers:

1. The temperature was for 1 hour and 25 minutes below 1°C
2. The temperature was from approximately $3h17'$ till $4h43'$ below 1°C

c. On the graph we read that the temperature drops to 0 to raise again afterwards. We calculate when $T(t) = 0$, for this the numerator of the fraction must be 0.

$$3t^2 - 6t + 3 = 0 \Leftrightarrow t^2 - 2t + 1 = 0 \Leftrightarrow (t-1)^2 = 0 \Leftrightarrow t = 1.$$

To algebraic calculate the whole exercise meaning that we don't use the graph, we have to check whether $T(t)$ has a minimum in 1. So we must calculate the derivative:

$$\begin{aligned} T'(t) &= \frac{(t^2 - 2t + 2) \cdot (6t - 6) - (2t - 2)(3t^2 - 6t + 3)}{(t^2 - 2t + 2)^2} \\ &= \frac{6(t^2 - 2t + 2) \cdot (t - 1) - 3(2t - 2)(t - 1)^2}{(t^2 - 2t + 2)^2} \\ &= \frac{3(t - 1)(2t^2 - 4t + 4 - (2t - 2)(t - 1))}{(t^2 - 2t + 2)^2} \\ &= \frac{3(t - 1)(2t^2 - 4t + 4 - 2t^2 + 2t + 2t - 2)}{(t^2 - 2t + 2)^2} = \frac{6(t - 1)}{(t^2 - 2t + 2)^2} \end{aligned}$$

$$T'(t) = 0 \Leftrightarrow \frac{6(t-1)}{(t^2 - 2t + 2)^2} = 0 \Leftrightarrow t = 1$$

To know whether this is indeed a minimum, we make a sign table of this derivative. The numerator is 0 if $t = 1$, the denominator is never zero because $D = 4 - 8 = -4 < 0$

t		1	
$T'(t)$	-	0	+
$T(t)$	$\searrow$	0	$\nearrow$

The algebraic calculation also shows that the temperature reaches a minimum of 0°C after 1 hour and starts rising from then on.

Answer: From 4h onwards, the temperature in the cool storage begins to rise again.

d. Graphically, we see that as t increases, the graph evolves towards 3. Algebraically, we calculate the limit for this as t goes to infinity.

$$\lim_{t \to +\infty} \frac{3t^2 - 6t + 3}{t^2 - 2t + 2} = \lim_{t \to +\infty} \frac{3t^2}{t^2} = \lim_{t \to +\infty} 3 = 3$$

Answer: the temperature in cool storage will be around 3°C over time.

Exercise C: The lawn of Mr. Jones'.

The lawn in Mr. Jones's garden measures 15 by 20 meters. Mr. Jones decides to extend the lawn. To two sides he adds a strip of equal width of x meters. See Figure 7.16.

a. Show that the area of the enlarged lawn is represented by *Area* = $x^2+35x+300$

b. The new lawn has an area of 374 m^2. Set up an equation and calculate the width of the strip.

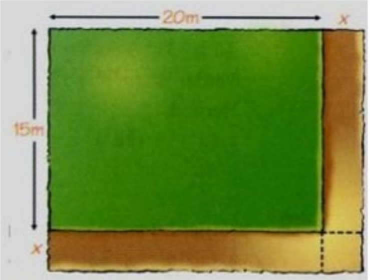


Fig. 7.16

Given:

- The lawn has a rectangular form with a width of 15m and a length of 20m.
- Mr. Jones enlarges his lawn with two strips of width xm both on the longest and shortest side of the rectangle.

a. The width of the new lawn is $15 + x$ and the length is $20 + x$. The area (A from 'area') is therefor $A(x) = (15 + x)(20 + x) = 300 + 15x + 20x + x^2 = x^2 + 35x + 300$.

b. If $x^2 + 35x + 300 = 374 \Rightarrow x^2 + 35x - 74 = 0$

$$D = 35^2 + 4 \cdot 74 = 1521$$

$$x = \frac{-35 \pm \sqrt{1521}}{2} = \frac{-35 \pm 39}{2} = 2 \text{ or} - 37$$

Because x is a real number, we reject the solution $x = -37$.

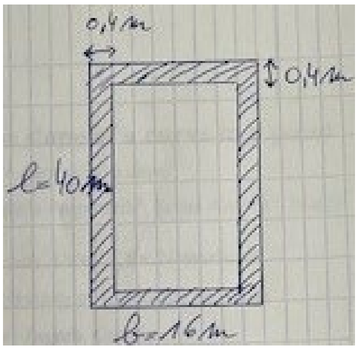
Answer: The width of the strip is $2m$.

Exercise D: Mr. Kok mows his lawn

Mr. Kok has a lawn of 16m by 40m. His lawnmower mows 40cm wide. He starts mowing on the outside and follows the perimeter. After how many laps is he halfway?

Given :

- The lawn of Mr. Kok has a width of 16m and a length of 40m and it is rectangular.
- The lawnmower mows according to a width of 40m



To the right, we first made a schets of this data:

To know when half of the lawn is mowed, we must know the area of the field. This is $16 \cdot 40 = 640$. Thus, half of this area equals $320m^2$

We must find when the lawn machine has mowed $320m^3$ because than the remaining part of the field to be mowed is $320m^3$ as well.

After lap 1 the lawnmower has mowed an area of: $0{,}4 \cdot 16 \cdot 2 + 0{,}4 \cdot (40 - 2 \cdot 0{,}4) \cdot 2$

The dimensions of the remaining part of the lawn are $b = 16 - 2 \cdot 0{,}4 = 15{,}2$ en $l = 40 - 2 \cdot 0{,}4 = 39{,}2$. Therefor its area equals $(16 - 2 \cdot 0{,}4) \cdot (40 - 2 \cdot 0{,}4) = (16 - 0{,}8) \cdot (40 - 0{,}8)$

We notice in this calculation of the area a link between lap 1 and the area. Therefor we go ahead with calculating per lap the area of the part of the lawn which is not mowed.

After lap 2 the dimensions of the port of the lawn which is not mowed is $b = 15{,}2 - 2 \cdot 0{,}4 = 14{,}4$ and $l = 39{,}2 - 2 \cdot 0{,}4 = 38{,}4$.

Let's analyse our calculations so far by writing down how we calculate, we need to find a general formula which gives a link between the number of laps and the area:

After lap 1 the area which is not mowed equals: $15{,}2 \cdot 39{,}2 = (16 - 0{,}8) \cdot (40 - 0{,}8) = (16 - 1 \cdot 0{,}8) \cdot (40 - 1 \cdot 0{,}8)$

After lap 2 the area which is not mowed equals: $14{,}4 \cdot 38{,}4 = (15{,}2 - 0{,}8) \cdot (39{,}2 - 0{,}8) = (16 - 0{,}8 - 0{,}8) \cdot (40 - 0{,}8 - 0{,}8) = (16 - 2 \cdot 0{,}8) \cdot (40 - 2 \cdot 0{,}8)$

After lap x the area which is not mowed therefor equals $(16 - x \cdot 0{,}8) \cdot (40 - x \cdot 0{,}8)$. When is this area $320m^2$?

$$(16 - x \cdot 0{,}8) \cdot (40 - x \cdot 0{,}8) = 320 \Leftrightarrow 640 - 12{,}8x - 32x + 0{,}64x^2 = 320$$

$$\Leftrightarrow 0{,}64x^2 - 44{,}8x + 320 = 0$$

$$D = 44{,}8^2 - 819{,}2 = 1187{,}84$$

$$x = \frac{44{,}8 \pm \sqrt{1187{,}84}}{2 \cdot 0{,}64} = 61{,}93 \quad \text{of} \quad 8{,}07$$

The number of laps can't be 61,93. If we replacet his value for x in width and length both dimensions would be negative which is impossible: $16 - 61{,}93 \cdot 0{,}8 < 0$ en $40 - 61{,}93 \cdot 9{,}8 < 0$. We must therefore reject this solution. These two negative values multiplied yield as much as 320.

8,07 means that Mr. Kok must mow a bit more that 8 laps to be halfway. Consequently, we round up to 9 rounds.

Answer: Mr. Kok must mow 9 laps to be halfway mowing.