Continuous professional development

Course Mathematics

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Content

Introduction

This course was created in collaboration with my colleagues and former colleagues from the teacher training programme at the Antwerp School of Education.

I thank Johan Deprez and Filip Moons who will recognise texts they either wrote themselves or in cooperation with me.

I would like to thank Ellen Vandervieren for the input she gave through the content of her lectures.

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I would like to thank Laure Ninove for her contributions in revising annex 1 and for her assistance with the French translation of the text.

I would like to thank Nele Kempenaars for the idea of the course layout.

I would like to thank all participants and instructors of the ProTEEM mathematics modules for their enthusiasm, which inspired me to write this course that provides additional background to the workshop slides.

I most sincerely thank Tom Smits who is the driving force behind the ProTEEM project for which, among other things, this course was written.

Gilberte Verbeeck

Essen, 11 april 2025

1 The lesson plan

1.1 Assignment for the next module

Every participant must make a lesson plan with all materials needed for a mathematics didactics lesson that the participant will teach in his own context. The subject of the lesson must be om Maths or Maths didactics.

1.2 Doel van een lesvoorbereiding

'Lesson preparation aims to leave the teacher in a strong position for lesson delivery.' (Didactisch referentiekader, Meeus & Verbeeck, 2016, ACCO, p 23). 'The idea that lesson preparation would curtail spontaneity is completely untrue. On the contrary, the better the preparation, the more energy is available to the teacher during implementation to respond spontaneously to what presents itself. After all, lesson preparation is not a straitjacket to be anxiously executed in detail. A good teacher knows how to use his lesson preparation flexibly and respond to the unexpected opportunities and difficulties of the moment. Lesson preparation helps the teacher assess in advance what scenarios might occur and how best to respond to them. This frees up space for spontaneous hunches, which he may or may not follow. Lesson preparation aims to aim is to leave the teacher in a strong position when implementing the lesson: the better prepared, the more room for spontaneity!' (Algemene didactiek, Meeus, Tanghe & Verbeeck, 2020, Acco, p21)

1.3 The lesson plan template

Lesson plans come in various shapes and sizes. We agreed to work with the following template which we explain further.

Lesson plan from xx (Name of University)

Subject & duration of the lesson	Student level
Lesson topic	**Number of students**

Initial situation

Please use the model of ASoE. Complete the 5 categories with relevant information for you lessen. *Add previous knowledge for the lesson under the category Student.*

Lesson specific objectives

Number	Objective

Evaluation/assessment during this lesson

Objective (Nr.)	Which assessment?

Sources – References

Lesson outline

Time	Objective	Learning content	Teaching method – Activity – Grouping	Materials – Seating arrangement

Board layout

A. Initial situation

We agreed to work with the model developed by Mees W. in the course Didactisch referentiekader.

Category	Context Factors
Pupil	previous knowledge, study level, intelligence, motivation, fear of failure, physical condition, general well-being, socio-cultural diversity, gender, …
Peers	group size, group dynamics, bullying behavior, subcultures, …
Teacher	age, education, pedagogical attitude, teaching style, intelligence, motivation, stress sensitivity, physical condition, general well-being, socio-cultural background, gender, teaching experience, career phase, …
School	(subject) colleagues, management, school size, school organization and policy, pedagogical project, school location and architecture, infrastructure and resources, …
Parents	financial, practical, and educational support, participation in school policy, …
Circumstances	education policy, socio-economic situation, current events, weather conditions, …

B. Lesson specific objectives

The central question here is: what should the pupils be able to do by the end of this lesson? The lesson objectives form the basis for a test on this topic. Use Kilpatrick's model (see 3.2) to check if you are sufficiently varying your lesson objectives.

For example, if you have introduced a concept, you might expect that after the lesson, pupils can provide a definition of the concept or explain an aspect of the definition, give examples, explain why the concept is important, and possibly tell when and by whom the concept was first introduced, etc. If you are giving exercises on solving certain integrals of polynomial functions, you might expect that pupils can solve a specific integral algebraically and with a graphing calculator or computer, provide a graphical representation of the specific integral, determine an upper bound of a specific integral for a given area, etc. Make sure to formulate the lesson objectives in observable behavior. For example: Pupils explain why an exponential function is only defined for $a > 0$ en $a \neq 0$.

C. Evaluation/assessment during the lesson

In this section, the focus is on formative assessment that gauges whether the individual pupil achieves one or more lesson objectives. It is not necessary to strive for completeness. The aim is to use one or more very short evaluation forms that provide insight into the learning process of the individual pupil. Whole-class activities are not suitable for this purpose because the teacher does not get enough information about where each individual pupil stands. For example, in a traditional classroom discussion, the teacher gains insight into what a few pupils can do, but never into what each pupil in the class can do (especially if the class size is larger than 10). When pupils work on exercises individually, it requires a lot of concentration from the teacher to know to what extent each individual pupil is achieving the lesson objective with these exercises. Methods such as an intro or exit ticket, a quiz, or the Q&A (Question and Answer method) where each individual in the class is activated, are suitable to include as evaluation forms in this section.

D. Sources – references

Refer to books on mathematical content or didactics, courses, journals, or websites that you use in preparing and delivering the lesson. Artificial Intelligence (AI) can be a useful sparring partner in this process. When using AI, it is interesting to include the prompt in your reference.

Useful questions are: how is the topic treated in different (hand)books, what precedes and follows the lesson you have to give, do you know anything about the origin of the concepts covered in the lesson, are there applications, etc.

If you use slides, a worksheet, etc., in the lesson, mention whether you designed it yourself or adopted it from someone else or the internet. It is not a problem to use someone else's work in your lesson, but proper citation is necessary.

E. Lesson outline

There are no fixed rules that guarantee a lesson is didactically well-structured. Many elements play a role in the didactically responsible construction of a lesson. Below, we discuss several aspects you should or can consider. Many of the elements mentioned below also interact with each other.

Even though a lesson hour in e.g. a secondary school only lasts 40 to 50 minutes, it is impossible for pupils to remain continuously focused for the entire period. For this reason alone, it is necessary to divide the lesson into smaller parts, which we call sequences. A typical duration for such a sequence is about ten minutes. By structuring your lesson using sequences, you immediately create relief during the lesson: during a sequence, you expect pupils to be focused, and between sequences, the reins can be loosened a bit. Often, one specific objective is associated with one sequence. It is good practice to briefly evaluate at the end of a sequence whether pupils have achieved this objective. On the one hand, each sequence should form a small coherent whole, but on the other hand, the lesson sequences together should also form a larger whole. In other words, there should be a red thread running through the lesson. This way, you ensure that your lesson is well-structured. (And of course, there should also be a red thread running through the entire series of lessons on a particular subject…)

Working with sequences not only helps you structure your lesson from the perspective of the learning content. It can also support you in choosing appropriate teaching methods. Often, it is not a good idea to lecture for an entire lesson or to have pupils work in groups for an entire lesson. It is better to vary teaching methods during a lesson, in other words, to choose different teaching methods for different sequences.

You should build your lesson from small, coherent parts that together form a logical whole. When preparing and delivering your lesson, you should pay attention to both aspects. At the beginning of the lesson, you pick up the thread again. You place the lesson in the larger context. You can either at the beginning of the lesson or later in another sequence of the lesson check whether the pupils have the necessary prior knowledge (from the previous lesson, an earlier lesson, a previous school year) to follow the lesson or part of the lesson. You can optionally end each part with a question to see if the pupils have understood this lesson sequence. The beginning and end of a sequence are also the ideal moments to highlight the red thread in the lesson: what is the goal we have in mind, and where are we currently on our way to this goal? At the end of the lesson, you briefly review what has been learned.

In your lesson preparation, you should clearly indicate these sequences. Separate them, for example, with a horizontal line (e.g., problem situation to motivate a concept, examples, definition, exercises). Choose for each sequence an appropriate teaching method. Indicate in the form a specific objective what you expect pupils to be able to do at the end of the sequence and ideally how you will evaluate at the end of a sequence whether pupils have achieved the objective. Estimate the time you think you will need for the sequence. So for each sequence you can complete the five columns as follows:

- In the first column, you estimate the time.

- In the second column, you write the lesson objective number you are working on. You write the numbers spread out next to the appropriate learning content at the correct height. It is not intended to place all the numbers at the beginning of the lesson sequence.
- In the third column, you write the learning content covered during the lesson. This is what pupils need to know after the lesson. For a Q&A you note the model answer to your questions. If pupils need to do exercises, you solve all the exercises. Write these solutions in this column or refer to an appendix where you add the solutions of the exercises.
- In the fourth column, you write what you and your pupils do during the lesson (the teaching methods with instructions, two-step questions for a Q&A, or what you will do or ask if there is no answer or not the expected answer, help questions if pupils need to complete an assignment, or if you call pupils to the board, etc.) and the grouping form. In this column, also prepare questions and follow-up questions to coach pupils if necessary while solving exercises. Be prepared for difficulties that learners might have to solve the exercise.
- In the fifth column, you list the tools. In the didactics of mathematics or computer science, these are, for example, a smartboard with a digital classroom book, GeoGebra, worksheets, cards to divide groups, materials for activating teaching methods, a graphing calculator, laptop, mobile phone, etc.

2 Lesson outline: Didactic Structure of a Lesson

2.1 (Anti-)didactic inversion

"*A common theme of the greater part of my publications has been: change of perspective; in particular what I called inversion and conversion, a mathematical virtue, practised and cherished from olden times. No mathematical idea has ever been published in the way it was discovered. Techniques have been developed and are used, if a problem has been solved, to turn the solution procedure upside down, or if it is a larger complex of statements and theories, to turn definitions into propositions, and propositions into definitions, the hot invention into icy beauty. This then if it has affected teaching matter, is the didactical inversion, which as it happens may be anti-didactical. Rather than behaving antididactically, one should recognise that the young learner is entitled to recapitulate in a fashion the learning process of mankind. Not in the trivial manner of an abridged version, but equally we cannot require the new generation to start just at the point where their predecessors left off.*" (a quote from a Dutch mathematician and mathematics educator Hans Freudenthal, Didactical Phenomenology of Mathematical Structures, 1983)

The order in which a finished piece of mathematics is written down (for example, a proof of a property in an article or book) differs from the order in which this piece of mathematics was discovered. Often, the order is reversed. Hans Freudenthal also points out that the order in a finished mathematical theory is often not the best order to use when learning mathematics. Therefore, he calls this change in order (and more broadly: change in the organization of the theory) didactical inversion.

This idea from Hans Freudenthal encourages us to question the order used in a mathematical theory regarding its use as an order during the learning process. Applied concretely to lesson construction: it is quite possible that the order in which the learning content is written in the textbook is not the most suitable order to use during the lesson. This does not necessarily mean that the order from the textbook is bad: it may correspond to the order you use after the learning process to clearly present what you have learned (because an inversion also takes place after the learning process…). This remark certainly does not apply to all textbooks: some textbooks pay a lot of attention to using an order that you can adopt during lessons.

We have already given the example of the construction of a proof of a property: the order in which a proof is written down afterwards often differs from the order in which the steps were taken when constructing the proof. We now apply the idea of (anti-)didactic inversion to several other examples (assuming a somewhat caricatured representation of a 'text that sets out a mathematical theory'; in practice, few mathematical texts are fully constructed this way).

- In a text that sets out a mathematical theory, a definition of a concept is first given, and then examples are shown. The standard order in a didactic context is reversed: first, treat examples and often counterexamples. This way, the concept takes shape informally. The definition then formalizes what the pupils already know, and you can involve the pupils in formulating it.
- In a text that sets out a mathematical theory, a theorem is first formulated, then proven, and applied to concrete cases. It can also be different. In the textbook "De question en question" from French-speaking Belgium (annex 1), for example, you see that pupils first 'discover' the Pythagorean theorem for a right-angled isosceles triangle and then investigate for which types of triangles the theorem might be valid before the theorem is formulated in general.
- In a text that sets out a mathematical theory, the theory is first constructed, and the applications come later. In a didactic context, it is often good to start with an application and build the mathematics from an application. In the book "Wiskunde vanuit toepassingen" (Mathematics from Applications), this approach is used for matrix multiplication (annex 2). The two matrices have a concrete meaning: one matrix contains the unit prices of different products in various stores, and the other contains the quantities a family wants to buy of the different products. The question of

how much this family should pay in each store leads to the matrix product as defined in mathematics. The definition of the matrix product does not appear here as a deus ex machina but as a mathematical abstraction of a calculation method that shows its usefulness in practice. If you use a textbook where the theory is first systematically set out and the applications come later (or not…), it can be a good idea to use one of the exercises as an introductory example for constructing a lesson.

- Mathematics is a deductive science. This means that new knowledge is derived purely by logical reasoning from already acquired knowledge. It implies, among other things, that in mathematics as a finished theory, reasoning is strongly from abstract to concrete. However, inductive elements also play a significant role in discovering new mathematics. Conjectures (statements that are suspected to be provable) often arise from studying examples and are often 'tested' in special cases before a general proof or counterexample is found. In a didactic context, it is often good to go from concrete to abstract instead of the other way around. In a sense, this is the overarching idea in the previous three examples.

2.2 From concrete to abstract

In paragraph 2.1, 'from concrete to abstract' was already briefly mentioned, but here we elaborate on it further. Working from concrete to abstract is less evident than it might seem at first glance.

A. Special versus generic examples

The first point we want to make is that when preparing an abstract concept, a property, a method to solve a problem, … using concrete examples, it is important that all the essential facets of the abstract concept are present in the examples. We give an example.

During a teaching practice, a pupil wanted to introduce the formula that expresses the distance between two arbitrary points in space in terms of their coordinates relative to an orthonormal coordinate system. He prepared this by asking the pupils to calculate the distance from the origin to point Q with coordinates $(4, 3, 2)$ by applying the Pythagorean theorem twice. After this example, however, the pupils failed to find the general formula. The preparation was not good. The example is not only concrete but also special, for example, because one of the two points is the origin. A better assignment is to calculate the distance from point P with coordinates $(3, 1, 5)$ to Q by using the Pythagorean theorem twice. This is a generic example. It is no coincidence that the third coordinate of Q is smaller than that of P. This prepares for the absolute value bars in deriving the general formula.

We give another example. Suppose you want to have pupils derive the formula for the *n*-th term of a geometric sequence for the first time, e.g, the geometric sequence with initial term $t_1 = 3$ and ratio 2. Of course, you first ask the pupils for the value of t_2 and t_3, for example. Then it is good to ask, for example, how you find the value of t_{10}. Although it is still a term with a concrete rank number, the pupils must already go through the general reasoning (we have to multiply t_1 by 2 several times in succession, from rank number 1 to 10 gives 9 steps, …). Calculating t_{10} is therefor a generic concrete example. After that example, the pupils can easily find the formula for the *n*-th term.

The message we want to convey in this paragraph is that you should work with generic examples if you want to prepare abstraction successfully. Generic examples are examples in which all the essential facets of the abstract concept are present. However, we must immediately nuance this:

- Important mathematical concepts are often so rich that you cannot find all facets in one example. Think, for example, of the derivative. Here it is better to work with several different examples. You can find this, for example, in the text on the introduction of the concept of the derivative from the magazine "Uitwiskeling" (annex 3).

- Sometimes it is too difficult to start with a generic example immediately. Then you can start with an easy, special example and gradually work towards a generic example. In the example of the distance between two points in space, you can work with the following sub-assignments:
  a. Calculate the distance from the origin to point Q with coordinates $(4, 3, 2)$ by applying the Pythagorean theorem twice.
  b. Calculate the distance from point R with coordinates $(3, 1, 1)$ to Q in the same way.
  c. Calculate the distance from point P with coordinates $(3, 1, 5)$ to Q in the same way.

B. Concrete and abstract

Our plea in favour of the concrete should certainly not be understood as a call to limit yourself to dealing with concrete examples. On the contrary, the intention is, of course, that pupils build up general knowledge and skills that can be used in various situations. The examples should support the construction of this general and transferable knowledge and skills, but we should not make the mistake of assuming that abstraction and transfer from these examples *automatically* occur. On the contrary, numerous studies show that transfer happens only very slowly.

Often you can tell that a pupil has understood a method in general, for example because they can apply that method in all concrete examples, but that does not necessarily mean that they can formulate it in a mathematical correct abstract way.

After dealing with examples, there must be an explicit or abstraction phase, and the learned material should preferably be applied to new examples. It is important that the relationship between the abstract formulation and the concrete examples is clear to the pupils. This can be done, for example, by carrying out the abstraction together with the pupils using the Q&A (Question & Answer) teaching method.

C. Between concrete and abstract

Concrete and abstract can be seen as two extremes of a continuum. Between the extremes ‘very concrete’ and ‘very abstract’ are many situations that are ‘partly concrete and partly abstract’. We connect several remarks to this.

Sometimes it is advisable not to carry out the abstraction in one go but to increase the degree of abstraction in several steps. In a problem with parameters, it is advisable to introduce the parameters one by one.

What is concrete for one person may not be for another. For example, the number 5 is a concrete object for a secondary school pupil, while a toddler or preschooler is still in the process of abstracting this concept from their environment. As a pupil learns more mathematics, more and more mathematical objects become concrete, and thus the concrete examples need to be 'less concrete'.

Related to the above, it is important to keep in mind that as a math teacher, you are much further along in the process of abstraction than your pupils. This can lead you to perceive some situations as 'the same' that pupils experience as different. For example, consider multiplying a graph with respect to the horizontal axis. The effect of such multiplication is perceived very differently depending on whether the factor is positive or negative (whether or not it involves reflection) and whether the absolute value of the factor is greater or less than 1 (stretching versus compressing the graph). In your examples and exercises, you will need to incorporate sufficient variation in this area.

D. Not only examples, but also counterexamples.

When generalizing from examples (consciously or unconsciously), there is always the danger of overgeneralization. A well-known example is incorrect proportional reasoning: many people assume a proportionality between quantities when it is not the case. For instance, many people incorrectly think that the area of a square doubles when the side of the square doubles. This is one reason why it is useful to provide not only examples but also counterexamples. Counterexamples help to delineate the boundaries of the concept, method, etc., being studied.

3 Mathematical proficiency

3.1 Case

During the lessons about second-degree functions, Mrs. Peeters ensures that there are many exercises in which the pupils must make a sign chart, e.g.

- Make a sign chart to analyze the functions behavior if $f(x) = 2x^2 + 2x - 4$
- Method?
- Calculate zeros via discriminant, enter them in the sign table and enter characters.

Her approach seems to work, because the pupils can soon easily draw up sign charts of 2nd degree functions.

As **a homework task**, Mrs. Peeters asks to do the following exercise:
You are throwing a ball with a friend. You throw the ball according to the function $y = -x^2 + 5x$. *It ends up neatly in your friend's arms. How far apart are you?*

Almost **no one** in the class manages to complete the exercise. How come?

Problem in case study

Based on the exercises in class, pupils have **mainly learned to apply a fixed procedure/technique** (cf. making a sign chart).

Pupils need more than that in their homework. They first have to translate problem into a mathematical formula. They have to check whether it is a second-degree function. They have to choose a good solution strategy. They should feel like getting stuck into the problem. Etc.

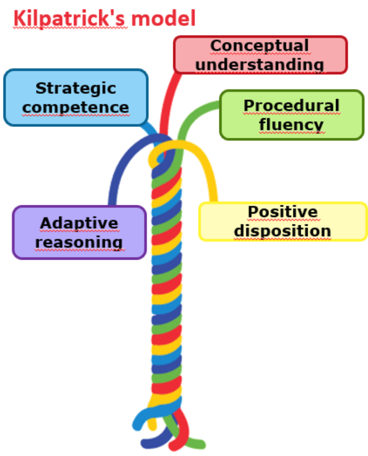
You need more than **procedural knowledge** to be able to handle mathematics smoothly (as a pupil and as a teacher!).

3.2 Goals of Mathematics Education

What is quality mathematics education? What goals must pupils achieve to attain a high level of mathematical proficiency? For generations, didactics experts have been seeking answers to these questions. In the United States, the National Research Council in Washington asked several prominent mathematicians, educational psychologists, educators, and mathematics didactics experts to define mathematical proficiency. This led to the publication of *Adding it up!* [Kilpatrick et al., 2001], which presents a model of mathematical proficiency that has since become a globally accepted model with five components which are strongly intertwined. When someone successfully practices mathematics, he needs them all. This interweaving is visualized by five strands that together form one thread (see figure below).

We describe the five components:

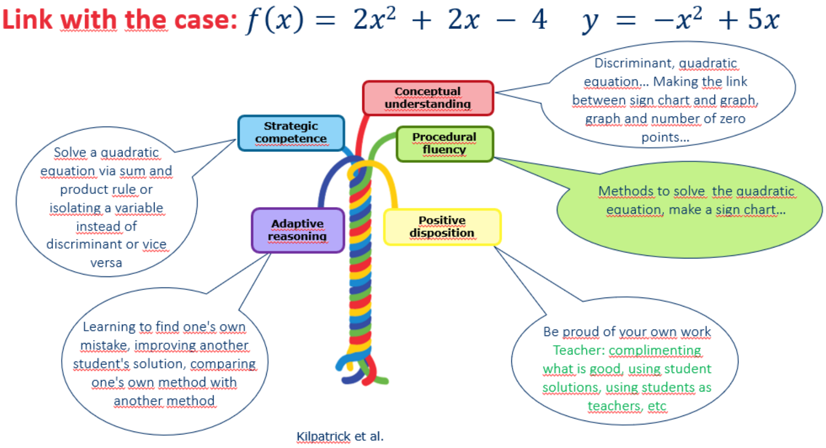
- **Conceptual understanding** = Understanding mathematical concepts, operations, and relationships.
- **Procedural fluency** = Knowledge of procedures and techniques to flexibly, accurately, and efficiently solve a certain type of (repetitive) exercises.
- **Strategic competence** = The ability to formulate, represent, and solve mathematical problems.
- **Adaptive reasoning** = The ability to reason logically, reflect, explain, and demonstrate.
- **Productive disposition** = Finding mathematics meaningful, useful, and worthwhile, coupled with the belief that effort pays off in becoming more mathematically proficient. Experiencing satisfaction and success in continually seeking solutions.



How do you translate these different components of mathematical proficiency into classroom practice? In the figure below, we provide an example of the various components related to quadratic functions and linked to the case in 3.1.

An attentive reader should notice two things: on the one hand, the model of mathematical proficiency evokes very little resistance; we all dream of making pupils competent in all these facets of quality mathematics education. On the other hand, we also observe that a lot of class time and evaluation focuses more than average on pupils' procedural knowledge. Take an average test or exam on quadratic functions, and most of the questions will assess procedural knowledge (e.g., solve this quadratic equation, conduct a sign analysis of the following quadratic function, etc.).

This discrepancy has also been extensively studied in the literature [Pesek & Kirshner, 2000]. One of the reasons for the above-average focus on procedural knowledge is because it is transparent and straightforward: pupils learn the solution strategy for solving a quadratic equation, and that is then asked on a test. Everyone is happy. Moreover, and not unimportantly, procedural knowledge is also crucial for the learning process. Without good routines, pupils struggle when deepening mathematical ideas or solving mathematical problems. The attention they need to work out their results instead of recalling them readily comes at the expense of attention to problem-solving approaches or seeking underlying relationships.

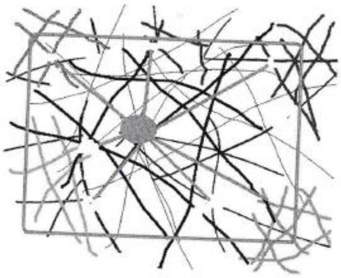


It is certainly not an either-or story: procedural knowledge is necessary, but if it is complemented with instruction and evaluation that also strongly focus on the other strands of the mathematical proficiency model, then learning mathematics becomes more sustainable and meaningful for pupils [Pesek & Kirshner, 2000]. Something you understand, you remember much better.

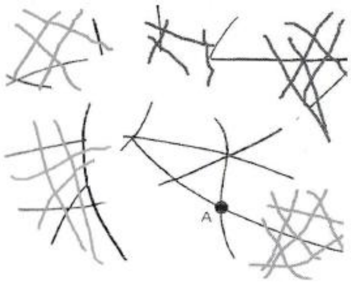
3.3 Cognitive Schemas

When pupils acquire procedural knowledge without a deep understanding of what they are doing, it leads to isolated chunks of knowledge. New concepts or skills cannot then build on an existing network of knowledge. In this way, pupils must learn new solution procedures for every small variation in problems.

One of the concepts to make pupils' conceptual knowledge visible is cognitive schemas. The term cognition stands for 'knowledge', for 'what someone knows', popularly for 'what someone's mental baggage is'. A cognitive schema is the coherent whole of someone's knowledge. Usually, it refers to knowledge in a specific area (e.g., exponential functions, limits, etc.). The term schema is used to clarify how one imagines the pupil's knowledge is organized (in memory).



As subject experts, we must be aware that we possess a particularly rich cognitive schema for almost all subject matter components. As the figure illustrates, we have an overview of all mathematics subject matter in secondary education and know the connections between a derivative, the slope coefficient, the difference quotient, limits, the relationship between rising/falling graphs...



On the other hand, our pupils, who are at the beginning of instruction, have a very poor cognitive schema. As this figure illustrates, they have not yet had the opportunity to make connections between different subject matter components, their knowledge is fragmented and not yet a coherent whole.

A. Assimilation

Cognitive schemas are not just static structures of stored knowledge; they are dynamic and change during the learning process. When a pupil adds new knowledge to an existing schema while learning, it is called assimilation. For example, a teacher can refer to the already existing knowledge of 'inverse operations' when teaching how to solve the equation $2^x = 7$. The inverse operation of 'plus' is 'minus', of 'times' is 'divided by', of 'square' is 'root', and similarly, of 'to the power of x' is 'logarithm'. A new concept is then integrated into an existing cognitive schema (assimilation), which becomes richer as a result. Naturally, it is not sufficient for the teacher or the textbook to mention this once; it is necessary for pupils to be able to explain it themselves at some point (=conceptual knowledge).

B. Accommodation

A cognitive schema can also change in a more significant way: sometimes it needs to be restructured. This is called accommodation. For example, when explaining how to solve linear equations of the form $2x + 7 = 30$ and $5x + 4 = 16$, a teacher might use the balance model. When equations of the form $2x + 21 = -7$ and $4 - 2x = x + 13$ also need to be solved, the balance model no longer works in an obvious way. Accommodating to a more formal level, the concept that the same operation must be applied to both sides, is then necessary.

C. Fragmentation

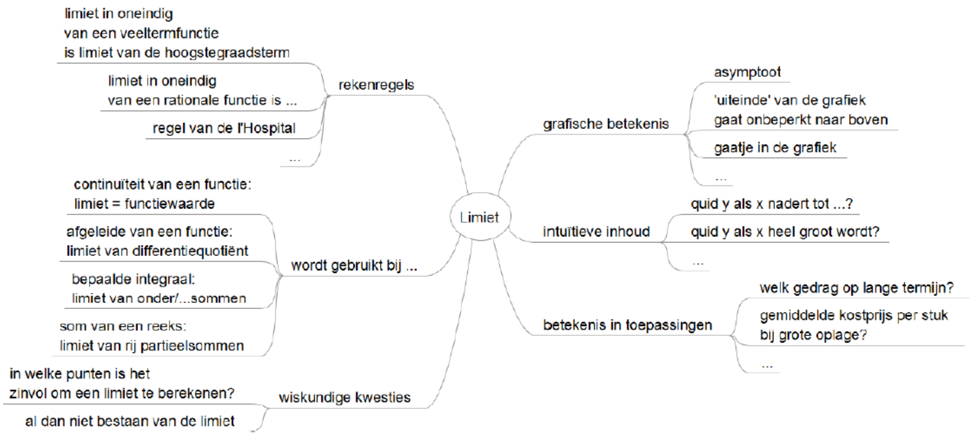
Van Dormolen [Van Dormolen, 1974] argues that breaking down the subject matter into small, manageable chunks is successful in the short term, but often leads to the storage of small, unconnected schemas in memory. This fragmentation results in pupils thinking they understand something (assimilating into an existing schema), while the schema is incomplete such as the ‘pupil version’ above. This is a real danger in education, where a strict curriculum must be followed, and the subject matter is already divided into manageable chunks per school year from above.

3.4 Role of the Teacher

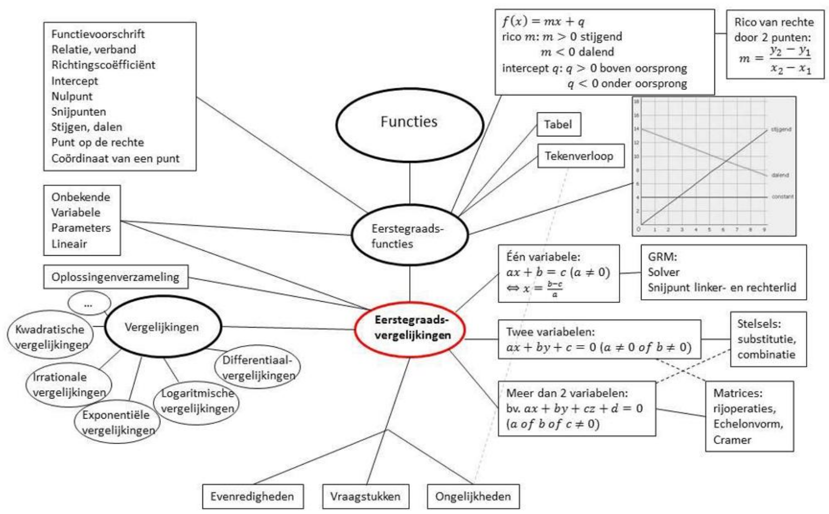
The cognitive schemas of pupils are particularly relevant for the teacher. To avoid falling prey to the so-called 'Curse of Knowledge' [Wieman, 2007], we must always and everywhere be aware that pupils' cognitive schemas are much more fragmented than our own. Every teacher can recall an insight they only gained when they started teaching certain subject matter. Well, pupils need to experience that every day, as they still have many insights to discover.

For the teacher, some connections are so obvious that the relationships are no longer explicitly explained to pupils. One way to address this is to create visual representations of the cognitive schemas to be achieved for subject matter components. The teacher can base the instruction on these. By visualizing the cognitive schemas, the teacher becomes much more aware of the necessary prior knowledge and the connections that need to be made. It is even possible to include the cognitive schemas in the pupil course materials or provide them to the pupils. Below are a few examples in Dutch.

Example 1: Limits Created by J. Deprez (KU Leuven) with Freemind
(see http://sourceforge.net/projects/freemind) [Deprez, 2017]



Example 2: Linear Functions Created by Jorinde Van Besauw (specific teacher training in mathematics, University of Antwerp)



References for the text in this paragraph (translated form the Dutch version written by Filip Moons):

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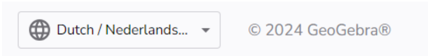
- Wieman, Carl. 2007. The "Curse of Knowledge" or Why Intuition About Teaching Often Fails. American Physical Society News.

4 GeoGebra

GeoGebra is a free online platform that gives a lot of free tools to do maths. It also connects enthusiastic teachers and pupils and offer them a new way to explore and learn about math. They can share their ideas and share the applets, worksheets… they made for their classes. It is useful both for teacher and pupils. It is extremely good to visualise mathematical concepts.

The GeoGebra Home page gives the overview of what you can do with GeoGebra. It's too much to list. Go to **www. geogebra.org** and set the language you are using to read the Home Page of the platform. Therefor scroll to the very bottom of the page and set your language.



The handout in annex 4 will let you explore the platform hands on.

5 Realistic mathematics education

5.1 Why should pupils learn math?

Depending on the pupil and his interest, we distinguish 3 reasons for learning mathematics.

1. Mathematics to use in everyday life

Most of what is really needed in everyday life are primary school subjects apart from statistics. Arithmetic or some basic elementary geometric knowledge is tackled at primary school. Some descriptive statistics or awareness of variability … is covered in secondary education.

Everyone, but especially young pupils and pupils who receive little mathematics, are the target group for this part of mathematics.

2. Mathematics for use in other disciplines

Mathematics supports the learning of concepts in other disciplines. Examples are: algebraic arithmetic, eigenvalues and eigenvectors, first degree functions, exponential functions, derivation and integration, differential equations, testing hypotheses, ...

The target group is future scientists (exact AND humane!), economists, statistics users, technicians, …

3. Mathematics for its own sake

Finally, there is mathematics for the beauty of mathematics itself. Pupils who show interest and talent in this should be taken through subjects such as: proofs, axiomatic and deductive work, mathematical structures, problem solving, beautiful geometric shapes, mathematics as a cultural object, ...

Obviously, the target group are future mathematicians and anyone who wants to be inspired by the way mathematics works.

5.2 What is Realistic Mathematics Education?

In Annex 5, you can read the following description of what we mean by Realistic Mathematics Education: '*Realistic Mathematics Education – hereafter abbreviated as RME – is a domain-specific instruction theory for mathematics, which has been developed in the Netherlands. Characteristic of RME is that rich, "realistic" situations are given a prominent position in the learning process. These situations serve as a source for initiating the development of mathematical concepts, tools, and procedures and as a context in which pupils can in a later stage apply their mathematical knowledge, which then gradually has become more formal and general and less context specific. Although "realistic" situations in the meaning of "real-world" situations are important in RME, "realistic" has a broader connotation here. It means pupils are offered problem situations which they can imagine. This interpretation of "realistic" traces back to the Dutch expression "zich REALISEren," meaning "to imagine." It is this emphasis on making something real in your mind that gave RME its name. Therefore, in RME, problems presented to pupils can come from the real world but also from the fantasy world of fairy tales, or the formal world of mathematics, as long as the problems are experientially real in the pupil's mind.*'

5.3 Authentic problems

A. Four Exercises

To tackle the problem of realistic mathematics education and mathematical proficiency, we investigate four exercises A, B, C and D. We first solve them to really get into the problems themselves. Then we analyse how the tasks themselves fit the concept of RME.

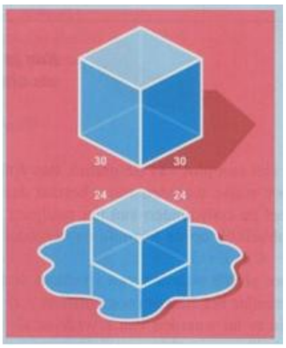
To analyse the 4 exercises, we present the following questions:

1. Arrange exercises A, B, C and D of most to least appropriate to boost mathematical proficiency (see paragraph 3.2).
2. Why do you think that assignment ... will boost the mathematical proficiency of pupils
    - the most?
    - the least?

Exercise A

An ice cube with edges of 30 mm long starts to melt down slowly. Every minute, the edges get 1.5 mm shorter. The volume of the ice cube is described by the formula
$V = (30 - 1{,}5\,t)^3$, where V stands for the volume in mm^3 and t for the time in minutes.

a. Calculate the volume of the ice cube when t=0.
b. What are meaningful values for t? And for V?
c. Plot and sketch that part of the graph for which the variables are meaningful.
d. Trace the graph with the cursor and investigate after how many minutes the volume is less than 10 000 mm^3. Provide your answer with a precision of one decimal.



Exercise B

The temperature in a cool storage room is given by the following function:

$$T(t) = \frac{3t^2 - 6t + 3}{t^2 - 2t + 2}$$

T = temperature (°C); t = time (in hours); t = 0 corresponds to 3 a.m.

a. Sketch the graph of this function
b. If the temperature drops below 1 °C, there is a risk of damage to the food. How long was the temperature below 1 °C? From when to when?
c. When did the temperature start to rise again?
d. What temperature does the cool storage evolve to?

Exercise C

The lawn in Mr. Jones's garden measures 15 by 20 meters. Mr. Jones decides to extend the lawn. To two sides he adds a strip of equal width of x meters. See Figure 7.16.

a. Show that the area of the enlarged lawn is represented by $Area = x^2 + 35x + 300$
b. The new lawn has an area of 374 m^2.
   Set up an equation and calculate the width of the strip.

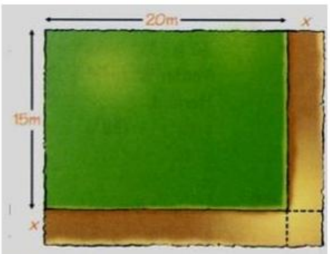


Fig. 7.16

Exercise D

Mr. Kok has a lawn of 16m by 40m. His lawn machine mows 40cm wide. He starts mowing on the outside and follows the perimeter. After how many laps is he halfway?

B. Solution of exercises

We include the solutions of the exercises in annex 6.

C. Conclusion

If we analyse the different tasks, we conclude to rank the exercises from D over B to A and C:

- We can call exercises like A and C 'Packed' exercises. The one that developed the exercise started from a bare mathematical exercise and came up with an application-oriented cover for it. This kind of exercises in no way shows how mathematics is applied in reality. On the contrary, it gives the impression that mathematics does not arise at all from problems one may encounter in life. It does not contribute to a better understanding of reality.
- Exercise B is a bit better. The function is a good model for the temperature in a cold room.
- Exercise D is a more authentic problems although it is still simplified.

Some notes on this task:

During the ProTEEM workshop an interesting discussion on exercise C about the context of extending the lawn came up. In the context of farming in Belgium, it is hardly possible that a farmer suddenly decides to extend his field since most (if not all) the farming land is owned by someone. Where in Zambia a farmer could have a plot and decide to bring one part of it in culture and wait for some time to bring more land in culture. So in this kind of context an exercise as C becomes more realistic.

A second note is that we usually need piecewice functions (the function has different expressions for different parts of the domain.") to model a realistic problem. However, for many secondary school pupils, tasks with piecewise functions are a bridge too far.

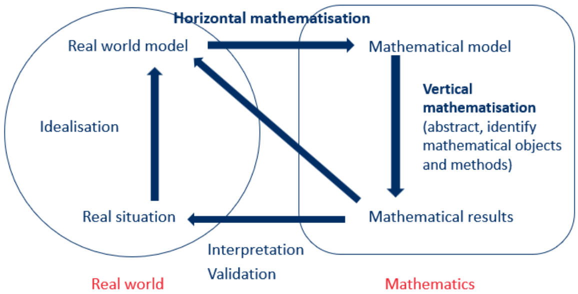
Authentic problems should be regularly integrated into a math lessons. They are necessary to allow pupils to experience the subtle relationship between reality and mathematics and to learn to deal with it (e.g. critically considering the solution). They allow pupils to see the ubiquity, applicability and beauty of mathematics and motivate them in the long run. Even if they are too difficult to solve, it is interesting to show pupils where maths is used for. We simplify exercises to make maths accessible at secondary level since the knowledge at this level is not sufficient to solve this type of real problems.

5.4 The ubiquity, applicability and beauty of mathematics

We can shape education so that pupils learn about the ubiquity, applicability and beauty of mathematics in the following

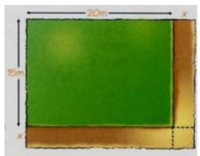
- Find problems/phenomena that require a mathematical approach, where mathematics really helps ("Didactic phenomenology")
- Connect mathematical trains of thought with phenomena in the physical, social, mental, ... world of pupils.
- Make pupils realize how mathematics can help to organize and structure real-world problems
- As a teacher, take into account the skills, competencies and interests (world) of pupils

We emphasize the importance of the **Reality Principle** (one of the basic principles of Realistic Mathematics Education). Pupils should experience assignments as 'realistic'. What is 'realistic' depends on age, prior knowledge..., in short the initial situation of pupils. The mathematical phenomena must connect to the pupils' world (i.e. real world, fantasy world, mathematical world) so that they can attach meaning to them and realize what they are doing. In addition, the tasks should be feasible in educational practice so that pupils feel competent to do the task. This is the C, an important pillar of the three universal psychological basic needs, the ABC which are key to the long-term motivation, personal growth and emotional well-being of youngsters: Autonomy, Belongingness and Competence.



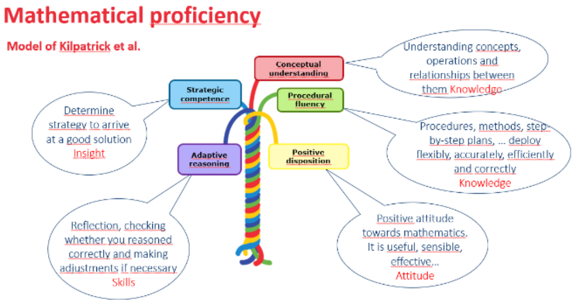
If we use real world examples, the issue of mathematical modelling comes up. Gabriele Kaiser developed the modelling cycle when modelling between the real world and mathematics. The scheme on the right shows the procedure. We start from a real-world situation. The first step is to understand the problem which means Pupils must identify and comprehend the real-world problem. Then this situation is idealised (see the arrow which points up in the circle), i.e. simplified or structured to get a real-world model. Pupils must make assumptions and simplify the problem to make it manageable. Then this real-world model is mathematised, i.e. translated into mathematical terms so that it leads to a mathematical model of the original situation. We talk about 'horizontal mathematisation'. The mathematical model is now a problem that can be solved using appropriate mathematical methods. Pupils must use mathematical concepts, procedures, methods, trains of thought to solve the mathematical problem which produces mathematical results (see the arrow which points down in the rectangular shape). We talk about 'vertical mathematisation'. Those mathematic results must be reinterpreted into the real situation. Pupils have to translate the mathematical solution back into the context of the real-world problem. And last but not least, the adequacy of the results must be checked, i.e. validated. In the case of an unsatisfactory problem solution, which happens quite frequently in practice, this process must be iterated. So Pupils have to check if the solution makes sense in the real-world context and refining the model if necessary (the diagonal and horizontal arrows pointing from mathematics to real world.

We place the exercises C and D within Kaisers model and check whether Pupils go through all steps of the modelling cycle.
In Exercise D the real-world situation is that we want to mow a lawn. The real-world model chooses the shape of the lawn to be a rectangle and that the lawnmower, can mow perfectly with right angles and follow the sides of consecutive rectangles each time... The mathematical model is a sketch of the lawn with all given information marked on it. During the vertical mathematisation we define the variable x (as the number of rounds the lawnmower must do), we use the formula for the area of a rectangle, we use the given information to create a quadratic equation and solve it. This gives us mathematical results which we must interpret and validate. In exercise D we have to exclude the solution $x = 61{,}93$ since this creates a conflict with the dimensions of the lawn.



In exercise C the start with the real-world situation is missing and the horizontal mathematisation is already done since the figure gives the mathematical model with even de choice of the variable x.
In exercise D this horizontal mathematisation is part of the exercise to be done by the Pupil. This is a step a lot of Pupils find difficult. If they can't exercise enough on horizontal mathematisation, they will never learn to do it themselves.

Therefore, teachers should pay attention to exercises in which both horizontal and vertical mathematization take place. In contexts where homework can be given to Pupils, we even advise to give enough time in class to work on horizontal mathematization and to leave the vertical one which has a lot to do with following procedures for the homework.



This benefits the mathematical proficiency of Pupils.

We place the exercises C and D next to Kilpatrick's model. In exercise C Pupils mainly train their procedural fluency. Strategic competence, adaptive reasoning and positive disposition is trained but to a lesser extent. Exercise D relies much more on strategic competence, adaptive reasoning, conceptual understanding and positive disposition than exercise C.

We conclude this chapter with an advice to enhance the verbal competences of Pupils: Let them communicate the results and the process to the class or part of it. The chapter is of course closely related to the next chapter on Problem Solving.

We refer to the article in annex 5 (van de Heuvel-Panhuizen, M., & Drijvers, P. (2014) form the *Encyclopedia of Mathematics Education,* p.521-525) to read more on Realistic Mathematics Education. In that same annex Tom Lowie writes on p.532-534 about 'Rural and remote mathematics education'. It might be interesting for our ProTEEM participants. It mentions communication technologies as a possible opportunity for distance education. In this way both chapter 4 on GeoGebra and chapter 9 on ICT integration might be useful to explore for a distance education set up.

6 Problem Solving

6.1 Experience what a problem is.

Mathematics teachers often don't realise (anymore) what a real problem is especially when they teach similar subjects and give similar exercises for a long time. They are so used to the exercises they cover with their pupils in class, whose answers they know almost by heart, that they rarely, if ever, experience how to tackle a new problem. That is why we start the lesson on Problem solving by presenting a problem that hopefully no one or enough students don't know. After all, a problem is a task, question, exercise… for which one does not immediately know how to find a solution.

A. The problem

The instruction is the following. The students work during 20 minutes in pairs. One person solves the given problem below. The other person observes and writes down which mathematical reflexes and methods are being used to solve the problem.

The problem:

The floor of a rectangular room is covered with (entire) square tiles. The room is m tiles wide and n tiles long. Half of the tiles are on the edge. For how many room sizes is this possible?

(A) none (B) 1 (C) 2 (D) 3 (E) more than 3

(Reference: Flemish Mathematics Olympiad volume 1996, 2e round, question 18)

The observer gets the following list of possible mathematical reflexes and skills (feel free to further supplement the list)

1. Working with concrete cases: reasoning based on an example.
2. Predict. At the start of the problem, you can think about which 'mathematical object' the result should be: a number with or without units, a ratio, a function, a yes/no answer, a proof... You can refer to this when formulating the decision.
3. Check whether your findings are not too specific (e.g. only for the chosen example).
4. A property is not valid as soon as you can give one counterexample. A property is only valid if it applies in all situations.
5. When checking a property, you should do this with different types of examples.
6. Checking examples to demonstrate a statement is not enough. In a further phase you also need general proof.
7. Generalize by introducing variables.
8. Building logical reasoning.
9. Search for errors and analyze them.
10. Check whether I am doing the right thing by looking back: does my approach lead to the answer to the question?
11. Be critical: Have I covered all options? Can I find an explanation for the results found? Are the formulas used correct?
12. Drawing up a table.
13. Interpretation of an answer (can I have all possible values for a number, does this function exist everywhere…).
14. Using symmetry.
15. Use of spatial insight.
16. Consider what knowledge I can use: When browsing through your knowledge, you think about which parts of mathematics you can use to solve the problem (algebra, geometry, analysis, etc.) and more specifically which theorems, properties you can use (congruent triangles, Pythagoras, solving a system, making a graph, solving inequality...).
17. You can first tackle a simple version of the problem, then try to generalize it.
18. Reverse reasoning.

19. When drawing up a proof, the following technique may help you further: 'start from the given, start from what needs to be proven, work in both directions at the same time, until both arguments 'meet' each other'.
20. A drawing (on paper or with a dynamic drawing package) is not a proof. If you draw a geometric situation accurately and as generally as possible, you can extract clues from it for a proof.
21. Using materials: sometimes you can try geometric problems with paper and scissors.
22. Making forecasts.
23. Working psychologically (whether or not to have confidence in what you are doing).
24. Correct notations are important in reporting: $3 + 2 = 5 + 7 = 12$ is an example of what not to do. Check your notations.
25. Heuristics: drawing a picture or making a figure, entering good notations, structuring the elements of the problem, knowing and looking at a related problem, analyzing data/request, wording the problem differently, which data are redundant, solving part of the problem, working form back to front, considering first a special case, looking at extreme cases, translating the problem into a more suitable representation, dropping a condition…
26. Try and error.
27. …

B. Solution of the problem

We include the solution of the problem in annex 7.

6.2 How to teach pupils to solve a problem?

The main part of the lesson is firstly a discussion on how the students arrive (or not arrive) at the solution. Because 20 minutes is very short to tackle a real problem. It is more likely that the duo's did not find a solution. But the question is how the solver proceeded. What did the observer notice? Which heuristics did the student use when trying to solve the problem. Which problem solving skills were observed? Secondly a brainstorm over the question 'What is essential when solving problems?' brings together the students ideas based on their experiences?

According to Schoenfeld (Reference: *A. H. Schoenfeld, Mathematical problem solving, Academic press, 1985)* the following skills or conditions are needed to be able to solve a problem:

- Knowledge base: one needs a substantive background (resources), i.e. knowledge of definitions, properties, techniques and frequently used methods, but also insight and intuition into the learning material involved.
- Problem solving heuristics. This are (internal) advices, methods or ideas that increases the chance of finding the solution, but without guarantee of success. They are **general**, meaning that they do not relate to one specific piece of learning material. In point 25 in the list in paragraph A the most important heuristics are given.
- Monitoring and control (Metacognition): A problem solver needs to manage the solution process agile. Rather than work like a chicken without head, one needs to plan, do an interim evaluation during the process, make decisions to e.g. start afresh… In this way, he actually has to string together the heuristics used and his substantive background. As mentioned above, heuristics are general and they have to be interpreted differently and appropriately for each problem. The list of heuristics is broad and not all heuristics are interesting to use for each problem. Selecting the good ones is a competence which is highly link to mathematical knowledge.
- Beliefs and feelings: They play a roll in the heads of the problem solver. Beliefs or opinions about the usefulness of mathematics and what is learned in previous maths lessons as an aid in searching for a solution for a problem. E.g. if pupils learned about the congruence cases of triangles in the lesson, they might not think that they can be an aid in searching for a problem, so that it is useful

mathematical knowledge even long after when they actually got to know them in the lessons. Previous knowledge not only for recent lessons but also for lessons in previous years is therefor important to be activated when solving problems. Belief or opinions about what is expected of pupils is another important item that plays a role in problem solving. E.g. giving up searching too quickly because of the belief that 'finding something yourself = genius'. Quite a few people think mathematics is only for geniuses and don't even start reading carefully as soon as they know it is a mathematical problem. This correlates strongly with pupils' feelings. How discouraged does one get 'if it doesn't work immediately', how confidence is one in his own abilities?

Until the 1980's one thought that only being competent in mathematical knowledge and procedures and the use of heuristics was sufficient to be able to solve problems efficiently. The book 'How to solve it' from Polya (1945) mentioned a set of heuristics. Research showed the importance of metacognition and beliefs. Schoenfelds theory dates form 1985, the model of Kilpatrick from 2001.

Schoenfelds theory is in line with the model of Kilpatrick we tackled in chapter 3. Having a good knowledge base refers to the strands of conceptual understanding and procedural fluency. The competency to use heuristics is a skill in line with the strategic competence strand. Monitoring and controlling the solution process is exactly what adaptive reasoning means. And lastly, beliefs and feelings fall within the strand of productive disposition.

What does this mean for maths lessons and teachers? Learning how to solve real (new) problems takes time and is possible to some extent. We can't expect that every pupil (and even not every teacher) will be able to solve real difficult problems, problems mathematicians solve to support the society we live in, the real-world problems. But teachers can bear in mind the following guidelines:

- Provide feasible, challenging assignments by sufficiently differentiating!
- Emphasize heuristics when they are used
- Provide exercises whereby students have to use monitoring and controlling skills. E.g. provide solutions of problems whereby students have to check what is wrong, why steps are right…
- Take the views and feelings into account! Views and feelings can change due to experiences in which pupils grow in self-confidence, e.g by giving attention to the usability of mathematics, by ensuring that pupils can gain success experiences in solving problems so that they realise that they are able to solve problems themselves, by learning that mathematics is not "finished" and that every search process is valuable and can lead to new, interesting things.

We introduce the concept of open and closed questions to give ideas on how to differentiate.

6.3 Open and closed questions.

A. Examples

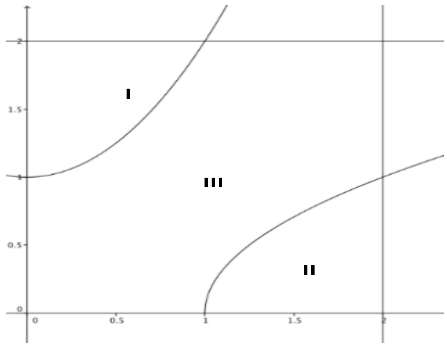
We look at the following exercise which can be solved in various ways. That is why it is very interesting to present it as an open question to the class. It is a question where only vertical mathematisation takes place.

We formulate it in an open and a closed question.

Exercise as an open question

In the figure you see the graphs of $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-1}$ and the lines $x = 2$ and $y = 2$. Three areas can be distinguished: I, II and III. Determine the area of each of the three areas I, II and III in (at least) two different ways.

Be creative!



Exercise revised into a closed question

In the figure you see the graphs of $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-1}$ and the lines $x = 2$ and $y = 2$. Whitin these formulas, three aras can be distignguished within the square formed by the positive x-axis and the y-axis and the lines $x = 2$ and $y = 2$: I, II and III.

a) Argue that the area size of area I is equal to the area size of area II.

b) Determine the area of each of the three areas I, II and III in (at least) 2 different ways. Be creative!

Tip: use your answer to question a) and properties of functions such as inverse function, symmetry, etc.

There are various ways to present these exercises in class. We give a few possibilities:

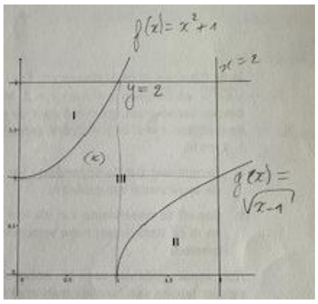
1. Divide the class in 2 groups (or more groups depending on the number of pupils). Give half the number of groups either the open or the closed question.
2. Divide the class in pairs. Each pair can decide whether to solve the open or closed question.
3. For the closed question: You can give the tip straight away or let students that struggle ask for a tip when needed.
4. Let students that solved the closed question present their solutions.

Solution

Since area I = area II we only have to find one of the two areas. This can be done in various ways. E.g.

Area of I = Area of a rectangle with width 1 and length 1 - $\int_0^1 x^2\,dx = 2 - \frac{x^3}{3}\Big]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$. So area I= area II = $\frac{2}{3}$

Area III = Area of the square with side 2 – I – II = $4 - 2 \cdot \frac{2}{3} = \frac{8}{3}$



In this solution, we shift the parabola $y = x^2$ and the straight line $y = 2$ one unit down on the coordinate system to calculate the area. We choose to calculate area I, since integrating a quadratic function is easier than integrating an irrational function.

A second possibility to calculate area I is by using the formula for an area between 2 graphs.

$$\int_0^1 (2 - x^2 + 1)\,dx = \int_0^1 (1 - x^2)\,dx = 1 - \frac{1}{3} = \frac{2}{3}$$

These 2 methods seem to us the easiest once. All other division or the areas into pieces will lead to more complicated integrals.

To the guidelines in the previous paragraph, we add the following:

- Provide feasible, challenging assignments by sufficiently differentiating! Design problems with open vs. closed formulation.
- Provide a closed variant of assignment for students whose problem-solving skills have not yet been developed strongly enough.
- Give students enough time to explore and form there “mathematical trains of thought” (see 5.4), to the logical progression of ideas and reasoning steps that a person follows when solving a mathematical problem or exploring a mathematical concept. Again for each exercise presented in class this involves:
  1. **Identifying the problem**: Understanding what is being asked.
  2. **Gathering information**: Collecting relevant data, formulas, and known values.
  3. **Formulating a plan**: Deciding on the methods and steps needed to solve the problem.
  4. **Executing the plan**: Carrying out calculations and applying mathematical principles.
  5. **Reviewing the solution**: Checking the results for accuracy and consistency.

  For example, if you're solving a quadratic equation, your train of thought might include recognizing the standard form of the equation, deciding how to solve the equation (use the quadratic formula or the method using sum and product formulas or a direct method when b or c are zero), when using the quadratic formula substituting the coefficients into the formula, solving for the roots, and then verifying the solutions.

  What we notice is that often steps 1,2, 3 and 5 are done with the whole class. Students can than only execute the plan which is using their conceptual and strategic knowledge. So even in the vertical mathematization we advise to number 4 for the homework.

Reference: *A. H. Schoenfeld, Mathematical problem solving, Academic press, 1985*

7 Teaching methods

7.1 Which Teaching Method?

Depending on the initial situation, lesson objectives, and learning content, one teaching method may be more suitable than another. Additionally, Pupils differ in the way they best acquire the material (their learning style) and which methods work for them. Therefore, it is important to vary teaching methods.

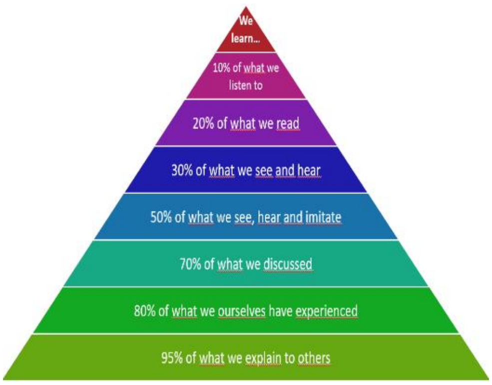
From Meeus, W. & Verbeeck, G. (2016). Didactisch referentiekader (2nd edition). ACCO pp. 95-96, we highlight the following quotes:

- ‘Activating education is education that succeeds in bringing Pupils to actively process the material.’
- ‘Activating education is education that encourages active learning, where material is acquired at a deeper level and remains stored in the brain for longer. This is called the didactic principle of activity.’
- ‘The degree of activation that the teacher can achieve through a teaching method is primarily related to the quality of their didactics.’
- ‘Every teaching method has the potential to promote active learning, just as every method can also remain at the level of busywork.’

The following schema is useful for assessing the quality of didactics and determining whether the activity is worthwhile. If Pupils enjoy playing a game but learn nothing substantive, we cannot speak of a good activating teaching method.

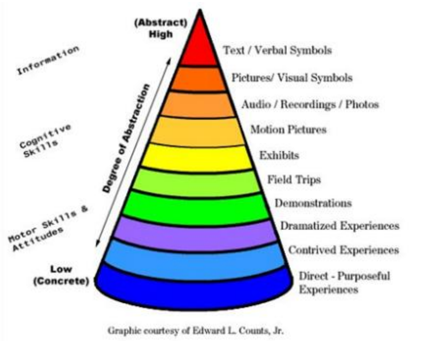
Activating education ⟷ Non- Activating education

Deep Learning ⟷ Surface Learning



The schema to the left is known as the learning pyramid of Bales. It is not a result of any research but based on personal experiences (subject to one's own learning style), we present it as a model to reflect on the didactics used in the lesson. However, both the content and the figures of this model are completely fabricated. It originated from a cone Edgar Dale presented in 1946 but it has taken a life on its own.

For your information, we provide the origin: an experience cone by Edgar Dale in 1946, summarizing and organizing various types of indirect learning from concrete to abstract. The cone has taken on a life of its own and resulted it the above.



7.2 Lecturing

"Lecturing is a teaching method that dates back to when books were so valuable that they had to be chained in the library. At that time, it was a necessity to transfer knowledge via the 'medium' of the teacher." (T. Geerligs & T. van der Veen)

Lecturing is a method where the teacher (lecturer) verbally conveys information or knowledge to a group of Pupils. The communication is thus one-sided, from the teacher to the Pupils, without interaction. A knowledge clip where the teacher explains a specific topic also falls under the lecturing method. The only difference is that pupils can pause and rewatch the video themselves.

This method has several advantages if executed well. The teacher can convey knowledge in a structured way in a short time. It is therefore very suitable for quickly offering the same information to pupils. It is also a very safe method. The teacher can prepare everything in detail. Due to the lack of interaction, the likelihood of unforeseen circumstances in the lesson is limited. This is naturally related to good classroom management where the group is quiet while the teacher lectures. With one-way communication, the teacher has everything under control and can work very structured. This method is easily usable for large groups. It can be a suitable method to explain a specific piece of theory, to make a proof and explain all steps, or to show how the teacher solves an exercise. The thought process of the specialist can be interesting for pupils in this last case. A caveat is that the teacher should not assume that pupils will simply copy what has been demonstrated.

The biggest disadvantage of this method is that due to the lack of interaction, it is almost impossible to find out if there are pupils who did not understand or missed parts. The teacher depends on a subjective interpretation of pupils' behaviour. Pupils' attention can also fluctuate greatly, and they may disengage. This attention typically wanes after 10 minutes of lecturing, and for some even faster. Many pupils ultimately remember information obtained through lecturing worse in the long term than information acquired more actively. Since understanding the material is crucial for a math lesson, lecturing is not an efficient method for most pupils, especially not for less motivated ones.

Points of Attention and Pitfalls:

- Ensure a very good structure. Regularly sketch the framework of the discourse and explain everything within it.
- Master the content thoroughly and with the necessary depth.
- Practice the lecturing moment to explain all transitions clearly and distinctly.
- Uphold the didactic principle of 'visualization' and support the explanation visually. Put a diagram or key words or formulas on the board or projection,
- The duration of a lecturing moment depends on the level and age of the pupils. 5 to 7 minutes is a good guideline for secondary education. In a 45-minute lesson, we advise that the teacher speaks for a maximum of 1.5 minutes per year of the pupils' age.
- Empathize with your pupils when preparing: how can you capture and maintain their attention? Anticipate possible problems for the pupils during preparation.
- Bring out your acting talent. An enthusiastic teacher holds attention much more easily than someone who does not seem to believe in his own lesson or subject. Never say you are presenting a boring piece of the subject.
- Keep an eye on pupils' body language. Pupils who break eye contact are usually disengaging.
- Check if pupils meet the following expectations: are they listening well? Are they taking good notes? Can they process the material independently after the lecture so that they understand everything?

7.3 Question and Answer method (Q&A)

"A Q&A Conversation is a pre-structured conversation by the teacher where pupils are gradually brought to certain insights through question and answer." (Meeus, W. & Verbeeck, G. (2016). Didactisch referentiekader (2nd edition), ACCO p. 81). A successful Instructional Conversation stands or falls with good preparation and guidance by the teacher. "Three requirements must be met: good questions; good questioning techniques; good handling of answers." (p. 81-82) Furthermore, "creating a positive climate" and "involving all pupils" are skills that a beginning teacher often has not yet mastered.

A. Good Questions

Think about the formulation of good questions in advance, write them in the lesson structure of the lesson preparation form. It forms the preparation of the method (Q&A) as a teaching and learning activity.

- Prepare crucial questions for steps/concepts that pupils find difficult. These are the hardest questions to prepare. For this, you must have thoroughly researched the content.
- Questions must be sufficiently specific. That is, questions that indicate a direction in which the pupil should look for an answer. Vague questions are, for example: "What does this mean? What do you see? What does this remind you of?"
- Prepare clear thinking questions that pupils can answer because they have the necessary prior knowledge.
- Work with a two-step model: start with a somewhat more difficult question (core question) and keep 'help questions' in reserve. If pupils do not know the answer to the difficult question, you can give them a nudge in this way.
- Formulate the correct answer you expect each time (both for core and help questions).
- Think about what wrong answers pupils might give for each question and how you will respond to these mistakes.
- Think about how you will respond if no pupil can answer your question.
- Do not ask questions where pupils must guess.
- Do not ask yes/no questions.

The following skills are trained during teaching. By briefly reflecting after each lesson, one will evolve faster.

B. Good Questioning Techniques

- Give pupils time to think about a question. You must be able to tolerate silence for this. Count to 10 in your head, this gives an indication of 10 seconds.
- Do not fish for one answer. The instructional conversation should not become a 'guess what word I have in my head'. Be sufficiently flexible with (subject) terminology and provide the correct (answer) word yourself if you notice that pupils understand the concept but do not use the correct word or formulate the answer. Repeating sufficiently will support them in this learning process of mathematical terminology.
- Questions like "Are there any questions?" or "Did you understand everything?" rarely lead to the behavior you want from your pupils. Usually, there is no response, which does not mean that pupils have no questions and understand everything. Ask content-related questions such as: "Explain this step.", "What is the explanation for this step?", "Which method did we use here?", "Why is this method a good method?", "How did you start this exercise? Which technique did you use for this?"…

C. Good Handling of Answers

- Do not show (e.g. by your facial expression) whether an answer is right or wrong, especially if you want to discuss it in class with the pupils.
- Have a pupil explain how they arrived at the answer.
- Possibly bring different answers to the same question to the board.
- Concisely write good elements from a pupil's answer on the board. This visualizes pupils' contributions, allows you to gradually build a correct answer, and supports the structure of the conversation.
- Do not systematically supplement minimal or incomplete answers from the pupil yourself. Involve other pupils where possible.
- Do not provide a corrected form of an incorrect answer. Have it corrected by other pupils.

D. Involving All Pupils

- Pass on answers and/or questions from pupils to other pupils. By passing a question or answer back to the class, you involve more pupils in the conversation. Avoid doing this only with incorrect answers. Pupils will see passing back as a hidden way of saying the answer was wrong. By passing back, you avoid pupils thinking "if we stay quiet for a while, the teacher will take over from us."
- Ensure that all pupils have the opportunity to answer.
- Ensure that the whole class hears the answer.
- Do not communicate with one pupil separately. Always keep the whole class involved.
- Give some thinking time after asking a question (count to 10) before designating the first pupil to answer. If you designate a pupil first, others are less likely to think along.
- Have pupils discuss briefly in pairs after you ask a question. This is called the think-pair-share method.
- Have your pupils write down an answer to a question before they speak.
- Interrupt the Q&A with tasks that you think pupils can handle independently. This is certainly the case with, for example, routine work.

E. Creating a Positive Climate Where Mistakes Are Allowed

- Do not always respond yourself: ask other pupils if they agree with the answer.
- Compliment a pupil who gives a (good) answer. A wrong answer can also lead to an interesting lesson activity.
- Use a good answer that deviates from what you had planned as much as possible. If you do not use it: try to explain why not, clearly indicating that the pupil's answer was also good.
- Criticize a wrong answer, not the pupil who made the mistake.
- Point out good elements in a wrong answer. Use the wrong answer in your instructional conversation as much as possible. The pupil may be making a mistake that many pupils make. Use this moment to zoom in on the mistake itself: explain why it is wrong, provide a possible cause for the mistake... For this, insight into the pupils' prior knowledge is needed.
- Ask the pupil to explain or clarify their answer. This way, the pupil sometimes discovers their own mistake (have the pupil present the answer on the board).
- Take photos of mistakes you encounter. You can use them in assignments where you ask pupils to find and correct mistakes.

The question 'Did everyone understand?' is nothing more than a rhetorical question in most classroom situations. Ensuring that everyone has 'understood it' automatically leads you to differentiation. You realize how complex teaching is. Preparing a lesson is a form of puzzling. The teacher has many puzzle pieces that they can place and rearrange in a way suitable for the learning content for their target group.

F. Some examples of a preparation for a Q&A in the lessonplan

In what follows, you will read some examples of a Q&A with assignments. Although thinking about the correct answers and possible answers students can give is part of good lesson preparation, we have not included this fully in the examples. We mainly offer the (crucial) questions and here and there additional ('help') questions according to the 2-step model.

Example 1: derivability of functions

Problem: Investigate the derivability of the functions in the given x-value

$$f(x) = \begin{cases} 1 - x^2 & \text{als } x \leqslant 1 \\ x^2 - 1 & \text{als } x > 1 \end{cases} \quad \text{in } 1$$

OLG with assignments

1. How do you calculate $f(1)$? Show your calculation.
   Answer: $f(1) = 1 - 1^2 = 0$
   Possible wrong answer: $f(1) = 1^2 - 1 = 0$

   Task: The teacher divides the class into 2 groups. Group 1 calculates the left limit, group 2 calculates the right limit. The pupil/student who finishes first from both groups brings the solution on board.
   Answer:
2. How can you investigate whether the function is derivative in 1?
   Answer: the function must be continuous in 1 and the left and right derivatives must be equal.
   Possible wrong answer: do not state the continuity. Since this is a technique that used to be taught, teaching repetition is the fastest method to keep the pace in the lesson.
3. How do you check whether the function is continuous in 1?
   Answer: left and right limits must be equal
4. Task: The teacher divides the class into 2 groups. Group 1 calculates the left limit, group 2 calculates the right limit. The pupil/student who finishes first from both groups brings the solution on board.
   Answer: $\lim\limits_{\substack{x \to 1 \\ <}} f(x) = \lim\limits_{\substack{x \to 1 \\ <}} (1 - x^2) = 0$ The function is continuous in 1.
   $\lim\limits_{\substack{x \to 1 \\ >}} f(x) = \lim\limits_{\substack{x \to 1 \\ >}} (x^2 - 1) = 0$
5. How do you check whether the function is derivable in 1?
   Answer: left and right derivatives must be equal
6. Task: The teacher divides the class into 2 groups. Group 1 first calculates the left derivative and then the right derivative, group 2 first calculates the right derivative and then the left derivative. When everyone has finished the first derivative, one student brings the solution on board so that the teacher and the students who have finished can check both derivatives.
   Answer:

$$\begin{aligned} \lim_{\substack{x \to 1 \\ <}} \frac{f(x) - f(1)}{x - 1} &= \lim_{\substack{x \to 1 \\ <}} \frac{1 - x^2 - 0}{x - 1} \\ &= \lim_{\substack{x \to 1 \\ <}} \frac{1 - x^2}{x - 1} \\ &\overset{\frac{0}{0}}{=} \lim_{\substack{x \to 1 \\ <}} \frac{-\cancel{(x-1)}(x+1)}{\cancel{x-1}} \\ &= \lim_{\substack{x \to 1 \\ <}} (-x - 1) \\ &= -2 \end{aligned}$$

$$\begin{aligned} \lim_{\substack{x \to 1 \\ >}} \frac{f(x) - f(1)}{x - 1} &= \lim_{\substack{x \to 1 \\ >}} \frac{x^2 - 1 - 0}{x - 1} \\ &= \lim_{\substack{x \to 1 \\ >}} \frac{x^2 - 1}{x - 1} \\ &\overset{\frac{0}{0}}{=} \lim_{\substack{x \to 1 \\ >}} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} \\ &= \lim_{\substack{x \to 1 \\ >}} (x + 1) \\ &= 2 \end{aligned}$$

   Since left and right derivatives in 1 are different, the function is not derivable in 1.
   Questions to ask students who cannot start:
   a. Give the definition of the (left/right) derivative.

b. What is $f(x)$?
   Answer: for the left derivative $(x < 1)$ is $f(x) = 1 - x^2$, fort he right derivative $(x > 1)$ is $f(x) = x^2 - 1$
   Help question: $f(x)$ is given. Which function prescription should you use if $x < 1$ (respectively $x > 1$)
c. What is $f(1)$?
   Answer: (see above) $f(1) = 1 - 1^2 = 0$
d. How do you calculate the limit $\lim\limits_{\substack{x \to 1 \\ x<1}} \frac{1-x^2}{x-1}$
   Help questions: Substitute the x-value by, what do you get? How to calculate a limit for an indeterminacy $\frac{0}{0}$? Factorize $1 - x^2$ and substitue this into the numerator.
   Antwoord: $\lim\limits_{\substack{x \to 1 \\ x<1}} \frac{1-x^2}{x-1} = \lim\limits_{\substack{x \to 1 \\ x<1}} \frac{(1-x)(1+x)}{x-1}$
e. To calculate a limit at the indeterminacy $\frac{0}{0}$, we delete the factor causing the '0' in numerator and denominator. What can you delete here?
   Help questions: Is $1 - x = x - 1$? Replace the variable x by a number, what do you get in both members?
   Answers: No, e.g. $1 - 3 \neq 3 - 1$ since $-2 \neq 2$
f. How can you still make sure you can delete?
   Answer: $1 - x = -(x - 1)$ or $x - 1 = -(1 - x)$

   For students who do not see this, the teacher visualises as follows:

$$\begin{aligned} 1 - 3 &= -2 = -(3 - 1) \\ 1 - x &= -(x - 1) \end{aligned}$$

g. Fill in everything you know by now and complete the exercise.
   Answer: $\lim\limits_{\substack{x \to 1 \\ x<1}} \frac{1-x^2}{x-1} = \lim\limits_{\substack{x \to 1 \\ x<1}} \frac{(1-x)(1+x)}{x-1} = \lim\limits_{\substack{x \to 1 \\ x<1}} \frac{(1-x)(1+x)}{-(1-x)} = \lim\limits_{\substack{x \to 1 \\ x<1}} \frac{1+x}{-1} = -2$
   Or $\lim\limits_{\substack{x \to 1 \\ x<1}} \frac{1-x^2}{x-1} = \lim\limits_{\substack{x \to 1 \\ x<1}} \frac{(1-x)(1+x)}{x-1} = \lim\limits_{\substack{x \to 1 \\ x<1}} \frac{-(x-1)(1+x)}{x-1} = \lim\limits_{\substack{x \to 1 \\ x<1}} -(1 + x) = -2$

Remark 1:
This is a non-generic task to compute the function value $f(1)$ as this gives 0 in both function rules. This is relevant for this exercise because it results in continuity in 1.
In the next task is $f(-3) = 2 \cdot (-3) - 1 = -7$. Entering -3 in the first function prescription gives the wrong result for the function value of -3 namely -6

$$f(x) = \begin{cases} 2x & \text{als } x < -3 \\ 2x - 1 & \text{als } x \geqslant -3 \end{cases} \quad \text{in } -3$$

Remark 2:
In [Puzzles]() you read about a teaching method that can also be used for the assignment in 6. The teacher cuts the solution into puzzle pieces. The instruction is:
1. Put the puzzle pieces in the correct order, i.e. the one that calculates the left (respectively right) derivative.
2. Have your solution checked by the teacher. Write the correct solution in your notebook and note the justification for why the step is correct at each step.

Example 2: Review on polynomials

The teacher wants to refresh students' prior knowledge about polynomials. He asks the questions: 'Tell me what a polynomial is?' 'What does a term consist of?' The students do not answer, this is called non-response. The teacher needs to make his questions more concrete.

Q&A

1. Give an example of a polynomial
The teacher notes several student examples on the board. He continues working with 1 or more examples. If students cannot give an example, he gives an example himself. This example is preferably one from a previous lesson where polynomials (possibly also in exercises) were covered. That way, the repetition principle works best. The following questions belong to the chosen example (from students or teacher):
2. Give one term of this polynomial.
3. How many terms does this polynomial have?
4. What is the degree of this polynomial?
5. The teacher indicates a term and asks the question: What is the degree of this term?
6. What do you call the number before the power of x?

Since the questions of the Q&A above have very short answers, you can also use this Q&A with the activation techniques covered in 7.4 A (scribble sheets or mini whiteboards).

Example 3: Definition divisibility in $\mathbb{R}[x]$

The teacher wants to apply the definition given in the following picture of the textbook available to students. The book works from concrete to abstract by giving a before-and-against example of divisibility in $\mathbb{R}$. This is not developed into a lesson activity. In what follows, we give some designs for an Q&A with small assignments. Each time, the teacher akss a few pupils to answer each question in order to arrive at the correct answer with the class. Each pupil carries out the small assignments possibly on a scrap paper.

> 1 Definition
> To define the concept of divisibility and to present its properties and exercises more clearly, we use two new symbols.
> **Notation:**
> **|** (read: "divides")
> ∤ (read: "does not divide")
> **Examples:**
> 3 | 6 because 6 = 3 · 2
> 10 ∤ 25 because 25 ≠ 10 · 2 + 5
> Divisibility in $\mathbb{R}$ [x]
> **In words:** $A(x)$ is divisible by $D(x)$ if and only if the remainder of the Euclidean division of $A(x)$ by $D(x)$ is zero.
> **In symbols:** $D(x)/A(x) \Leftrightarrow \exists Q(x) \in \mathbb{R}[x]: A(x) = D(x) \cdot Q(x) \;\Leftrightarrow R(x) = 0$

From: Van Basis Tot Limiet 4 - Functies - D-finaliteit 5 uur – Chapter 3 Algebraïsch rekenen p139 ev (Publisher die Keure)

Note on this definition: the part about R(x) comes out of the blue. In symbols, a better definition would be:

$$D(x)/A(x) \Leftrightarrow \exists Q(x) \in \mathbb{R}[x]: A(x) = D(x) \cdot Q(x) \text{ in doing so } R(x) = 0$$

Q&A with small assignments 1

1. Give an example of an ascending division in $\mathbb{R}$.
Additional question: Give two numbers where one number divides the other.
2. Explain why your answer is correct.
Extra question: Explain why the number 3, divides the number 6 (we note this as 3/6).
3. For the statement $3/6$ what is the divisor and the divisor?
4. For the statement $6 = 3 \cdot 2$ what is the divisor, the dividend and the quotient (two possibilities for the divisor!)
5. Give two numbers that do not divide each other.
Extra question: Does 10 divide the number 25? Why yes/no?
6. Explain why 10 does not divide the number 25.
7. Write 25 as an expression of numbers containing the number 10.

8. In question 7, name the different numbers.
   Tip: use the names divisor, dividend, quotient, remainder.
9. Write the expression in question 7 with symbols, choose the letters you use such that there is a link to what they stand for.
   Answer: $a = d.q + r$
10. Give a general notation for divisor, dividend and quotient in polynomials.
    Answer: $A(x), D(x), Q(x)$ en $R(x)$
11. Under the statement $6 = 3 \cdot 2$ write down a similar statement for polynomials. Use the symbols from question 10.
12. Write a definition in symbols for an ascending division $D(x)/A(x)$

Q&A with small assignments 2

1.to 4. same questions as above

In a previous lesson, the following exercise on Euclidean division was presented:

$$A(x) = 2x^3 + 3x^2 - 8x + 3 \text{ by } D(x) = x - 1$$

$$\begin{array}{rrrr|rrr}
2x^3 & +3x^2 & -8x & +3 & x & -1 & \\ \cline{5-7}
-(2x^3 & -2x^2) & & & 2x^2 & +5x & -3 \\ \cline{1-2}
 & 5x^2 & -8x & & & & \\
 & -(5x^2 & -5x) & & & & \\ \cline{2-3}
 & & -3x & +3 & & & \\
 & & -(-3x & +3) & & & \\ \cline{3-4}
 & & & 0 & & &
\end{array}$$

$Q(x) = 2x^2 + 5x - 3$ en $R(x) = 0$

5. Why do we call this an exact division?
6. $3/6$. You used to learn that we then speak of an exact division because there is a number, namely 2 so that $3 \cdot 2 = 6$. Here, 2 is the divisor. For the division of the above polynomials, write down a statement such as $6 = 3 \cdot 2$ with the polynomials from the exercise.
7. Write a definition in symbols for an exact division $D(x)/A(x)$

Q&A with small assignments 3

1. In the examples or exercises from previous lesson, find a euclidean division where the remainder is 0.
2. For your example or exercise, write down what $A(x), D(x)$ en $Q(x)$ are.
3. Check that $D(x) \cdot Q(x) = A(x)$.
4. In such a case, we say that $D(x)|A(x)$, what does this mean in words?

A few examples from the students come to mind.

Q&A with small assignments 4

Do not start from a concrete example, but from abstract prior knowledge. Indeed, depending on the initial situation of the class, you can also start abstractly.

1. In a previous lesson, Euclidean division was defined as follows:

Euclidean Division Let $A(x)$, $D(x), Q(x), R(x) \in \mathbb{R}[x]$ and $D(x) \neq 0$ The Euclidean division of $A(x)$ by $D(x)$ is written as: $A(x) = D(x) \cdot Q(x) + R(x)$ with $\deg(R(x)) < \deg(D(x))$ or $R(x) = 0$. Here, Q(x) is the quotient and R(x) is the remainder.	From: Van Basis Tot Limiet 4 - Functies - D-finaliteit 5 uur – Chapter 3 Algebraïsch rekenen p139 ev (Publisher die Keure)

   What will be the statement for $A(x)$ if the remainder is the 0-polynomial.
2. We refer to such a situation as an exact division. Write a definition to accompany the statement: $A(x)$ is divisable by $D(x)$.

7.4 Activating pupils withing the Q&A teaching method

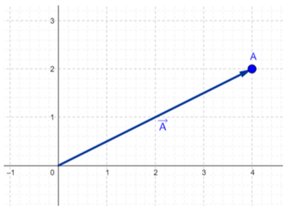
The Q&A teaching method has pitfalls. A teacher cannot always correctly interpret whether all pupils keep the attention to the lesson subject, especially not in large class groups. In what follows we present some ideas to prevent that pupils disengage.

A. Answers in the air

Every (pair of) pupils gets some draft sheets of paper with a thick felt-tip pen or mini whiteboards with a marker. When the teacher asks a question, pupils write the answer on their draft paper or mini whiteboard. The teacher gives a signal and the (pairs of) pupils raise the boards in the air. At a glance, the teacher can see what each pupil (or pair of pupils) answers. No pupil escapes the thinking activity. In the picture the pupils had to draw the graph of a straight-line meeting certain conditions.

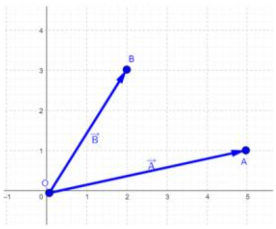
A variation on this same method is that the teacher prepares some coloured cards or cards with letters.

The teacher asks a multiple-choice question. The pupils look for the card with the correct answer. They raise the cards in the air after the teacher gives a signal. The examples below show two multiple-choice questions for both options



What is the coordinate of the point vector $\vec{A}$?

- **(2,4)**
- **(4,2)**
- **(2,1)**
- **(1,2)**

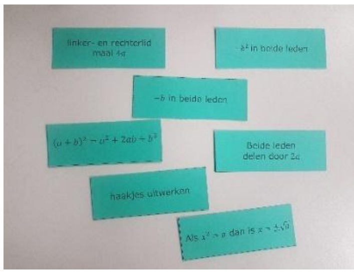


What is the coordinate of the point vector $\vec{A}+\vec{B}$?

A. (7,3)
B. (5,3)
C. (7,4)
D. (2,7)

Another variation on this same method is that the teacher prepares some cards with answers to questions, justifications of a step in a proof or a solution of an exercise or a justification/formula/definition/rule… to take a further step in a proof or a solution of an exercise.

The examples in the picture and table show some cards that can be used when proving a property. The cards show justifications for taking certain steps in the proof. Pupils think about a property or definition that can justify the next step given by the teacher or that can be used to take a next step in the proof. They raise the card in the air. Usually, pupils who choose the right card can give a conclusive justification (in correct mathematical language or not).



The picture shows cards used to proof the property on the solutions of a quadratic equation $ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

The table shows cards that can be used to proof a theorem of integral calculus.

Definition of derivative	Definition of integral function	Mean value theorem	Addition and subtraction
Extended definition of definite integral	Additivity of the definite integral	Calculation of limit	

B. Flash talks

Flash talks is a question-and-answer teaching method whereby the teacher ask questions, and randomly selects any pupil to answer the question. The teacher keeps the pace high, the tension in the learning conversation and does not give students the chance to wander off. A prerequisite for this is that the whole Q&A runs smoothly. It requires high concentration from the teacher. The questions must have the right level of difficulty and thus be solvable within seconds, otherwise the flow of the conversation will be lost and attention will wane. A more difficult question is possible, but then the teacher must give the pupils some time to think. About two minutes seems long to the teacher, but to think about a more difficult question it is very short. During flash talks it might be advisable to functionally walk around the classroom to see what pupils are doing and how they are thinking or to prompt them for activity. Flash talks work very good in strong classes. The quality of the questions is crucial and as said above the solvability of them is very important for this method to be used in a weak class.

7.5 Teaching methods: Groupwork

There are different forms of group work: parallel group work and group work with experts whether phased or not. More than ever a good preparation of the organisation is important for the groupwork to be activating or not. A smooth start is a prerequisite. First of all, a clear and complete instruction for pupils on what they exactly are required to do, is very important. Secondly this might include that the pupils quickly must change the arrangement of desks in the classroom into one that supports the implementation of the work form. Last but not least, ways of quickly form groups support a smooth start.

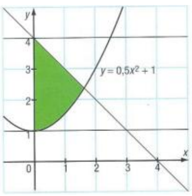
A. Parallel group work

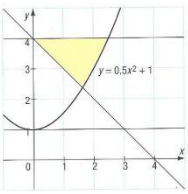
A parallel group work is one whereby each group is given the same assignment(s). This kind of group work is often done in classes where the teacher gives some exercises and the pupils work in pairs to solve them. After some time, the solutions the groups have are checked in class or the teacher makes answer keys available (on paper or online). It is advisable, depending on the level of difficulty of the exercise, to have a class discussion about the methods used, the answers found and about different (good and bad) answers and methods.

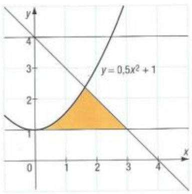
B. Phased group work with experts

Pupils work in different phases with differently composed groups. A common phasing is that pupils first work individually on a limited number of different assignments. Next, the pupils with the same assignment form expert groups in which they talk about the assignment together and thus become more skilled (expert) in solving it. The groups are then redistributed so that the mixed groups contain one pupil from each expert group. The information from the expert groups is thus brought together so that the mixed groups together have enough knowledge to handle a more complex task. The following is an example whereby different exercises are solved in different groups so not every pupil in class solves all the exercises himself.

The pupils get when entering the classroom, a paper with a graph of an area (coloured orange, yellow or green). The task is that pupils have to find the area using integral calculus. During a short first phase every pupil tries to solve the exercise on his own.







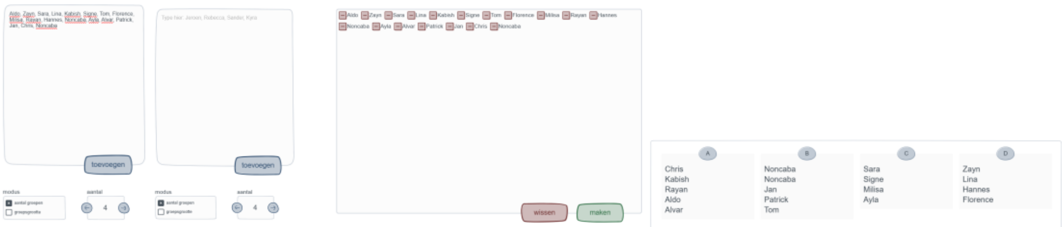
In phase 2 the pupils with the same-coloured area form a group. They put together what they found and thus become more skilled (expert) in solving the exercise. The groups decide on the best method and answer and write their best method and answer on a piece of paper. They hand it in to the teacher (either when finishing or at the end of the lessen).

In phase 3 the groups are redistributed in new groups with one or two pupils from each expert group. Each pupil presents the solution of his exercise. The other pupils listen and ask questions where they do not understand.

The teacher marks the solutions from the different groups (not from every individual pupil) and either discusses his comments in class or makes it available for pupils (online or on paper).

Variations:

- On phase 1: the pupils already tried all the exercises at home and the class time is used for discussions in the small groups. So only phase 2 and 3 are done in class
- To divide the class in groups:
  - The green area is somewhat easier to solve that the yellow and orange one.
    - The teacher let the pupils choose which area they want to solve.
    - The teacher let first all pupils solve the green area. After that he divides the class in 2 groups for them to solve the yellow and orange area.
  - Use 'groepjesmaker': **www.schoolbordportaal.nl/schoolborden/programma-839.html**. The screenshots below show how to enter names, add (toevoegen) them and form (4) groups.



- One phase 3: Each group presents their solution to the exercise for the rest of the class.

The following movie gives an idea of this activity in class: https://youtu.be/j93o26fIBG4. The video can be used for a Video-task: How active are the learners? Is deep learning taking place?

We notice that learners are active with the content and exercise since we notice that:

- Discussions are going on;
- Pupils give explanations to each other;
- Pupils experience success by finding the solution (hands in the air);
- Some pupils work individual within their group;
- The blackboard is used by various pupils to put different solutions (about the same exercise) next to each other;
- Pupils use their smartphone to take pictures of the blackboard.

The chairs and tables are arranged in L-form and in the middle in U-form. This is easy to quickly move from classwork to groupwork.

7.6 Teaching methods: Educational games

Games can inspire for teaching methods to teach, practise, or repeat certain learning content. Some mathematical games are commercially available, but they might seldom be exactly what the teacher wants to use in class. Whether you make your own game or buy one, it is important to set up or adapt the rules properly so that deep learning of mathematics takes place. As mentioned earlier, playing the game is not at all a goal. The pupils have to learn mathematical concepts, procedures, reflect on their work… We present a few different types of educational games.

7.6.1 Association games

Memory and the quartet game allows practising connections between two or four concepts. These two games are ideal for bringing concepts together which is seldom covered in exercises. Developing exercises where students put together matching concepts supports building a rich cognitive schema.

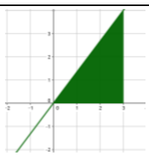
A. Memory

The goal of the game is that players turn over as many pairs of corresponding cards as possible. The material needed is an even number of cards whereby each two cards form a matching pair. The way to play the game is the following. The cards are shuffled and placed face down. The first player turns over a card of his choice. He then turns over a second card. If the pictures on both cards form a matching pair, the player can take the two cards and keep them until the game is over. He then takes another turn until he has turned over two cards that do not form a pair. The two face-up cards are put back in their place, covered. Now the turn goes to the next player. The game ends when all cards have been taken. The player with the most pairs wins.

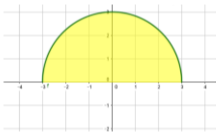
Memory is one of the games whereby the teacher must change the rules. Deep learning of mathematical concepts has nothing to do with remember where which cards I on the table, but it has to do with the matching pairs. Therefor one could say that during 3 to 5 minutes the pupils can play the real memory game. After that they have to turn all the cards so that the pictures are visible and then check as a group for the matching pairs.

We give a few examples to illustrate what you can use memory for: graphs with matching precepts, integrals with their geometric meaning, the left and right sides of a formula, two screens of a graphing calculator that belong together, powers with their corresponding integer exponent, powers with their elaborated form...

<u>Example 1</u>: Match the graph with the function rule



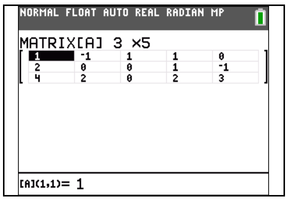
$$f(x) = \frac{4}{3}x$$

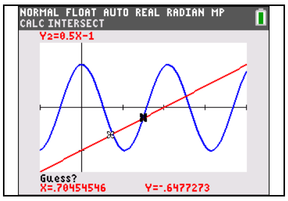


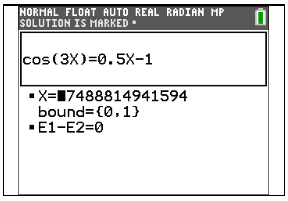
$$f(x) = \sqrt{9 - x^2}$$

<u>Example 2</u>: Match the corresponding calculator screens

$$\begin{cases} x_1 - x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_4 = -1 \\ 4x_1 + 2x_2 + 2x_4 = 3 \end{cases}$$



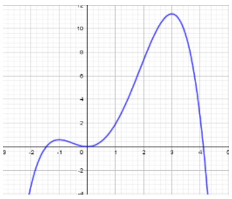


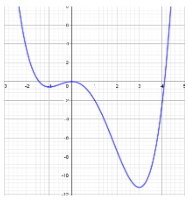


<u>Example 3</u>: match the sign chart to the corresponding graph

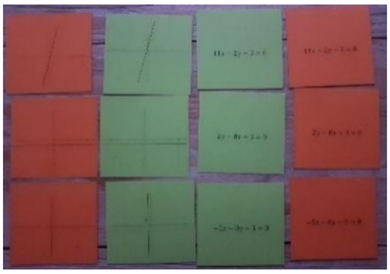
x		-1		0		3	
$f'(x)$	-	0	+	0	-	0	+

x		-1		0		3	
$f'(x)$	-	0	+	0	-	0	+





Example 4: This example is similar to example 1 but it gives an extra possibility. The memory cards are printed in 2 colours. These can also be used to form pairs and, over time, groups of 4. First, pupils with corresponding cards in the same colour form a pair and, after a while, the pairs of green and orange with the same corresponding cards join together to form a group of 4 pupils.



B. Quartet

A quartet is a dedicated deck card game. Each pack originally contained 32 cards, divided into 8 groups of 4 cards that belong together. Such a group of 4 cards are called a quartet. Over time the number of cards varied as long as the total number of cards is a multiple of 4. The goal of the quartet game is to collect as many quartets as possible. To play the game each player gets 4 cards. The remaining cards are placed face down in a pile in the middle of the table. Each player checks whether he has a quartet. If so, he places it face up in front of him and takes 4 additional cards from the pile. Player 1 draws a card from the remaining pile. He keeps it and puts one card face-up in the middle of the table for the others to see. He thus keeps 4 cards in his hand. If he has a quartet, he puts it open in front of him and takes 4 more cards from the pile. He checks again if this is a quartet. Player 2 starts. He may take player 1's face-up card or a card from the pile. Player 2 keeps the new card and puts one card face up on the on the table and a quartet if he has one. In this way the game continues until all the cards are gone.

Quartet is another game whereby the teacher should change the rules for it to be suitable for deep learning of mathematical concepts. Indeed, the game can go on for too long without collecting quartets. It can be e.g. that several players try to collect the same quartet. If no quartets are collected, the game defeats its mathematical purpose. After all, we want to collect 4 corresponding cards.

We deviate from the game rules and play the game in rounds with adapted rules. The following adaptation ensures that we slowly move away from the game and focus on the goal of 'building a rich cognitive scheme'. In round 1 each player plays one time. After that pairs (or in case of odd number duos and trios) are formed. In round 2 each pair (or trio) puts their cards together. Quartets they place open on the table. For each quartet the pair puts face-up, they take 4 additional cards from the pile or from cards lying face-up in front of other pairs. Then the pairs (or trios) play against each other as in round 1. In round 3 all players play together. They put all the cards face up on the table and make all the quartets.

Example 1: Volume and area

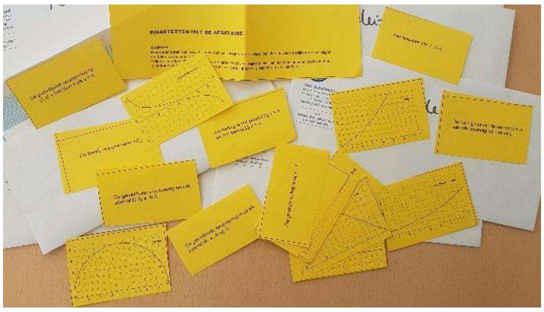
The 4 matching cards deal with the volume or the area (picture) and the equivalent formula.

Example 2: Exponential functions

The 4 matching cards give a formula, a sign chart, the growth factor (or percentage) and time (half-life or doubling).

Example 3: Derivatives

The 4 matching cards give the graph of the function, the derivative function, a description about the slope and the average rate of change.

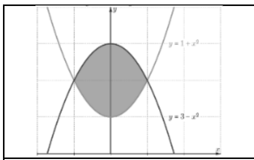


Example 4: Exercises in geometry

The 4 matching cards give the same result. Pupils solve the exercise on the cards. Those with the same answers form a quartet.

Example 5: Exercises on integral calculus

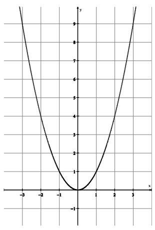
The 4 cards match with a figure and make pupils think about different task to solve including a step in the solution.

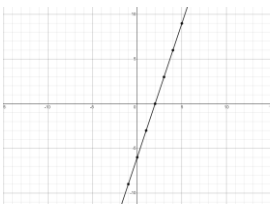
	I am finding either the area of the coloured surface or the volume when I rotate the surface about the x-axis
$\pi\int_{-1}^{1}[(3-x^2)^2-(1+x^2)^2]dx$	$2\pi\int_{0}^{1}((3-x^2)-(1+x^2))dx$

C. Associations with any number of matching cards

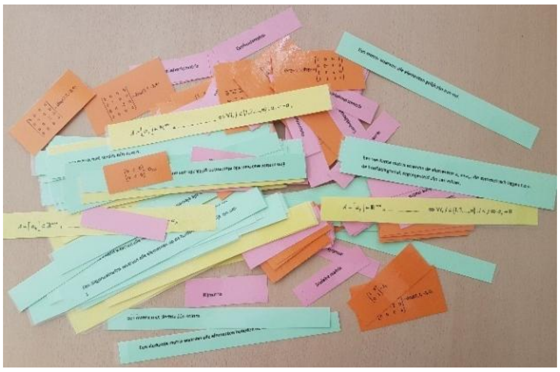
Of course we can make association cards with any number of cards. Find some more examples below.

Examples 1: They all deal with 3, 5 or 9 concepts for functions. The pictures show classroom use.

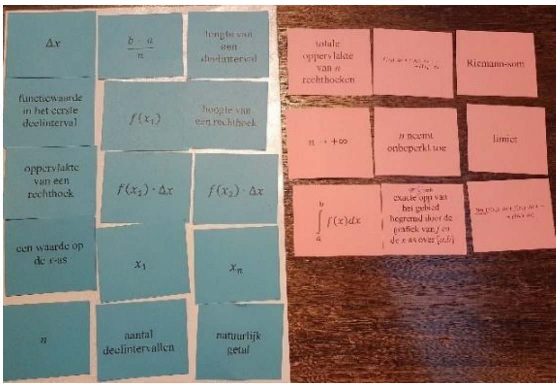
<table>
<tr>
<td>$y = x^2$</td>
<td></td>
<td>
<table>
<tr><th>x</th><th>$f(x)$</th></tr>
<tr><td>-3</td><td>9</td></tr>
<tr><td>-2</td><td>4</td></tr>
<tr><td>-1</td><td>1</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>4</td></tr>
<tr><td>3</td><td>9</td></tr>
</table>
</td>
<td>
<table>
<tr><td>x</td><td></td><td>0</td><td></td></tr>
<tr><td rowspan="2">$f(x)$</td><td>+</td><td>0</td><td>+</td></tr>
<tr><td>↘</td><td>min</td><td>↗</td></tr>
</table>
</td>
<td>$dom\ f = \mathbb{R}$
$ran\ f = \mathbb{R}^+$</td>
</tr>
</table>

<table>
<tr><td>$f(x) = 3x - 6$</td><td>The name of the function is f and the equation of the function is $y = 3x - 6$</td></tr>
<tr><td>It is a linear function</td><td>It is a first-degree function</td></tr>
<tr><td>(3,3) and (2,0) are elements of the function</td><td>The slope of the straight line is 3</td></tr>
<tr><td>The differential quotient of two points of the function is $\frac{9}{3}$</td><td>
<table>
<tr><td>x</td><td>-3</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>y</td><td>-15</td><td>-12</td><td>-9</td><td>-6</td><td>-3</td><td>0</td><td>3</td></tr>
</table>
</td></tr>
<tr><td></td><td></td></tr>
</table>

Example 2: In this example the pupils have to match all the cards to a certain special matrix: a name, a definition in words or in symbols, an example, a property… The number to be matched varies depending on the special matrix.



Example 3: In this examples all the matching cards are part of the definition of a definite integral. It is developed so that pupils show that they understand the different parts and elements in the definition. They have to match 3 cards. The blue cards are easier than the pink once. The whole class starts to make the blue association. Faster pupils can tackle the pink cards.



7.6.2 Domino

A domino game consists of several rectangular tiles (dominos), usually with a line dividing its face into two parts. The right part of one domino matches with a left part of another domino. The goal is to put the dominos together in the correct order. It has a similar mathematical use as memory: matching two concepts.

To play the game all the dominos are distributed among the participants. The participant who has the ‘startstone’ (e.g. ‘fundamental indefinite integrals’ or ‘goniometric START’) can start the game. If the next participant has a matching domino he can lay his domino, otherwise he gets a penalty point. The game continues like this until all stones are laid. The player with the fewest penalty points wins.

Of course one can skip the game part and just let individuals or groups of pupils lay all the dominoes in the correct order. In this way one forms a ‘snake’.

Example 1: formula (integral calculus and trigonometric)

Fundamental indefinite integrals	$\int x^r dx$	$\frac{x^{r+1}}{r+1}+c$	$\int \frac{dx}{\cos^2 x}$
$\tan x + c$	$\int \frac{dx}{x}$	$\ln\lvert x\rvert + c$	$\int \frac{dx}{1+x^2}$

Goniometric **START**	$\cos(\alpha-\beta)$	$\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta$	$\sin 2\alpha$
$2\sin\alpha\cos\alpha$	$\tan 2\alpha$	$\frac{2\tan\alpha}{1-\tan^2\alpha}$	$\sin^2\alpha$

Example 2: solutions of short exercises (remarkable products, derivatives, roots and powers)

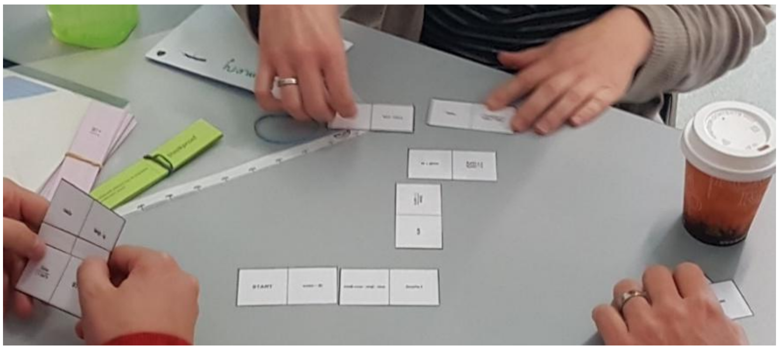
START	$(x-3x^2)(x+3x^2)$	x^2-9x^4	$(6a-4)^2$
$36a^2-48a+16$	$(a-b)(a+b)$	a^2-b^2	$25+100x+100x^2$

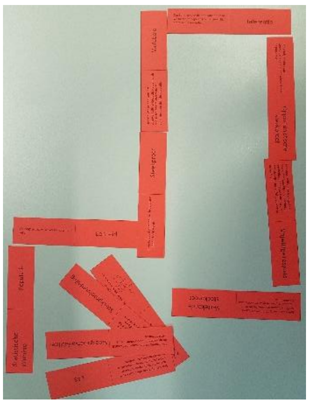
START	$D(2x^2)$	$4x$	$D(3\cos x)$
$-3\sin x$	$D(3-\sin x)$	$-\cos x$	$D(4x^2-5x+8)$

START	$\sqrt[3]{a^9b^{12}}$	a^3b^4	$\sqrt[5]{32a^{15}}$
$2a^3$	$\sqrt[6]{a^{6x}b^{12y}}$	a^xb^{2y}	$\sqrt[5]{a^{2^{20}}}$

Example 3: definitions (statistics)

Statistical domino	Population	Full group about which we want information	Unit
Each individual element of the population under investigation	Sample	That part of the population that is actually examined with the aim of obtaining reliable information on the whole population	Variable





7.6.3 Puzzles

Teaching pupils how to make a mathematical proof is not easy. Working with proofs individually and thus understand them or make them themselves is an important intermediate step. Making a puzzle is an example of such an step. The method can be used for proofs but also for solutions of exercises that might be difficult to find.

If one can prove a property or identity by proving the right hand side (RHS) from the left hand side (LHS) or the other way arround, the different steps can be ‘cut’. The pupil must make the puzzle by putting the different pieces (steps) in the right order. He should each time what the justification of a step is, why he can put a certain piece as the next step. A variation to this method is to make puzzle pieces of the justifications of steps as well.

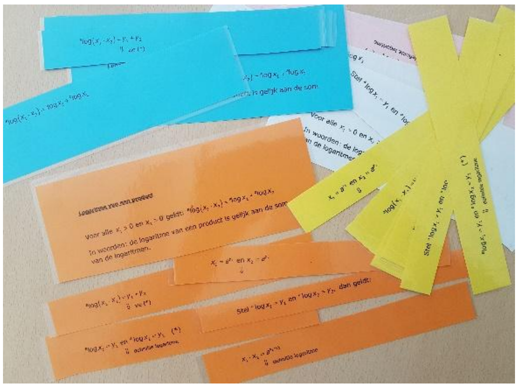
Example 1: main theorem of integral calculus

In the picture you find a few puzzle pieces of the proof.

$I_a'(x)$	$= \lim\limits_{h\to 0} \frac{I_a(x+h)-I_a(x)}{h}$	
$= \lim\limits_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)dt + \int_x^a f(t)dt\right)$		$= \lim\limits_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right)$

Example 2: the logarithm of a product

In the picture you find the same puzzle in different colours. This makes it easier for teachers to keep different puzzles of the same property separate.



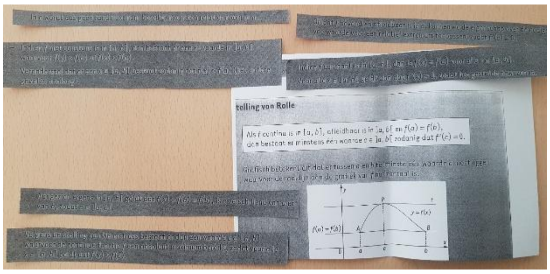
Example 3: Sum and product of the roots of a quadratic equation

The picture shows the puzzle pieces needed for both proofs. A variation (differentiation): Let pupils choose whether they want the pieces for the different proofs separate or together.

$\frac{-2b}{2a}$	$\frac{b^2-(b^2-4ac)}{4a^2}$	$\frac{4ac}{4a^2}$
$\frac{-b+\sqrt{D}}{2a}+\frac{-b-\sqrt{D}}{2a}$	$\frac{(-b)^2-(\sqrt{D})^2}{4a^2}$	$\frac{(-b+\sqrt{D})(-b-\sqrt{D})}{4a^2}$
$\frac{-b}{a}$	$\frac{b^2-b^2+4ac}{4a^2}$	$\frac{b^2-D}{4a^2}$
$\frac{-b+\sqrt{D}-b-\sqrt{D}}{2a}$	$\frac{-b+\sqrt{D}}{2a}\cdot\frac{-b-\sqrt{D}}{2a}$	$\frac{c}{a}$

Example 4: Roles property

The picture shows that the teacher copied from the textbook to make the puzzle. This reduces the work to make the puzzle to a minimum, but the exercise for the pupils remains the same.



Example 5: derivative of a^x

The picture shows the justifications for different steps in the proof.

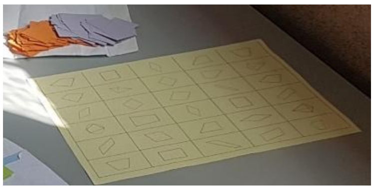
Definition derivative	$f(x) = a^x$	Property powers
$\lim\limits_{x\to a}(c.f(x)) = c.\lim\limits_{x\to a} f(x)$	$\lim\limits_{h\to 0}\frac{a^h-1}{h}$ is a number that varies with a	distributivity

Since, as so often in an exercise, not all students will finish at the same time, the teacher should think about tempo differentiation. Every pupil must do the first step, get enough time to think and make the puzzle. Those that are ready can either get cards with explanations or not (give pupils the choice) and justify each step. They can help others. They can write down the proof without the puzzle in their notebook and point out difficult steps. Pupils could also be given the choice at the beginning to make the proof with or without the puzzle etc.

7.6.4 Connect four

The material for this game is a board with a number of columns and rows and two stacks of cards each in a different colour. You play the game with 2 participants (or 2 groups) and every participant has a stack of cards in his own colour. In turn, each participant turns over a card from his stack and places it on the board. The goal is to be the first one to form a continuous row with 4 cards of one's own colour either horizontally, vertically or diagonally.

The example tackles quadrilaterals and their definitions and properties. The picture shows a grid to be cut with the quadrilaterals and the two stacks in different colours. The board below consists of descriptions of quadrilaterials. The players must put his card with a quadrilateral on a correct place on the board. The quadrilateral must match the description on the board.



rectangle	trapezium	quadrilateral with at least two equal sides	square	quadrilateral with perpendicular diagonals
rhombus	quadrilateral whose diagonals bisect eachother	quadrilateral with at least one pair of parallel sides	quadrilateral	quadrilateral with at most one pair of parallel sides
cyclic quadrilateral	isosceles trapezium	quadrilateral with at least two equal angles	rectangular trapezium	parallellogram
quadrilateral with four equal sides	non-rectagular trapezium	trapezium	parallellogram	quadrilateral with at least one obtuse angle
cyclic quadrilateral	rhombus	polygon with the sum of angles 360°	rectangle	quadrilateral with four equal angles
regular polygon	quadrilateral with equal diagonals	quadrilateral with at least two equal sides	trapezium that is not a parallelogram	quadrilateral with at least one right angle

7.6.5 Escape room

This teaching method is developed by Mirte Vangrunderbeek, a student teacher at our School of Education. Thanks to this teaching method, pupils make exercises on transition matrices very enthusiastically and are satisfied with the prize if they crack the code: the solutions of the exercises and a new task.

The material consists of cards for a group division and 3 worksheets with exercises and answer keys.

A. The group division

Upon entering the classroom, students are each given a card with an element of a matrix $[a_{ij}]$ on it. A matrix is projected on the blackboard. The example of such a table and matrix below is suitable for classes with 12 pupils or less.

a_{11}	a_{12}	a_{13}	a_{14}
a_{21}	a_{22}	a_{23}	a_{24}
a_{31}	a_{32}	a_{33}	a_{34}

$$\begin{bmatrix} 3 & 4 & 11 & 4 \\ 16 & -1 & -8 & 11 \\ -8 & 3 & 4 & -1 \end{bmatrix}$$

Students look for the number corresponding to their card in the projected matrix. Students with the same number form a group. If there are 11 pupils in the class, the teacher does not distribute a_{21}. Thus, a total of 5 groups form: 1 group of 3 (4), 4 groups of 2 (3,-8,11,-1).

B. Solving exercises via three worksheets

Each group of pupils works on the same worksheets. Each worksheet provides a code to go to the next worksheet. Pupils work 30' to crack all the escape room codes. The teacher projects a countdown clock.

The groups start the first worksheet at the same time. This contains two exercises on constructing transition matrices. If the transition matrices are correct, pupils find the code at the bottom of the worksheet to open a lock of a box. In that box, each group will find the correction key to worksheet 1 and the task from worksheet 2. The group that finishes worksheet 1 first takes an answer key and a new task, closes the lock and thus the box. They first correct worksheet 1 and show it to the teacher.

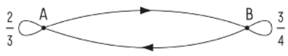
Then pupils make worksheet 2 and complete two exercises on drawing a graph corresponding a transition matrix. When their graphs are ready, they go around the classroom looking for the graphs corresponding to their solutions. On the walls are several papers, each showing two graphs with a QR code. There is one paper with the correct solutions, the other papers show wrong solutions. When the pupils find the paper that shows their combination of graphs, they scan the QR code. This gives the text 'Oops... these are not the right graphs' when the solution is wrong. 'Check your exercises again. You can do this!' For correct solutions, pupils read 'Congratulations, you've completed the exercises successfully. Look under your chair for worksheet 3!'

On the third worksheet, pupils complete an application exercise on the number of Belgians who would have lived or are living in the Brussels Capital Region in 2017 and 2030. On the back is a task to combine the numbers that come in the answers into new numbers. There is also a map of the class with some of the numbers resulting from the exercise. The correct solution provides a place that pupils discover via a map of the classroom. At this place, they find an envelope containing the correction key from worksheet 3 and a question from the test.

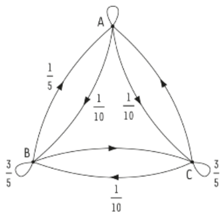
C. Worksheet 1

Problem: Complete the graph and note the corresponding transition matrix.

Exercise 1



Exercise 2



Code

Cijfer 1: $6.a_{11}\ (oef\ 1)$	**Cijfer 2:** $3.a_{21}\ (oef\ 1)$	**Cijfer 3:** $5.a_{22}\ (oef\ 2)$

D. Worksheet 2

Problem: Draw the graph for the following transition matrices.

Exercise 3

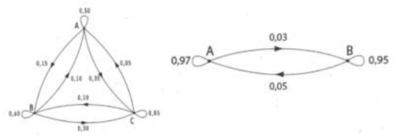
$$\begin{array}{cc} & \text{van} \\ & \begin{array}{cc} A & B \end{array} \\ \text{naar} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0{,}97 & 0{,}05 \\ 0{,}03 & 0{,}95 \end{bmatrix} \end{array}$$

Exercise 4

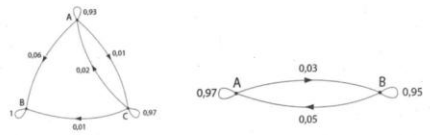
$$\begin{array}{cc} & \text{van} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{naar} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0{,}93 & 0 & 0{,}02 \\ 0{,}06 & 1 & 0{,}01 \\ 0{,}01 & 0 & 0{,}97 \end{bmatrix} \end{array}$$

Several graphs are hidden in the classroom. Find the right combination of graphs that match your solution and scan the corresponding QR code.

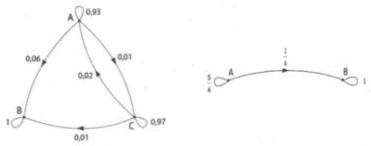
Worksheet 2: Some examples of answer keys



SCAN ME



SCAN ME



SCAN ME

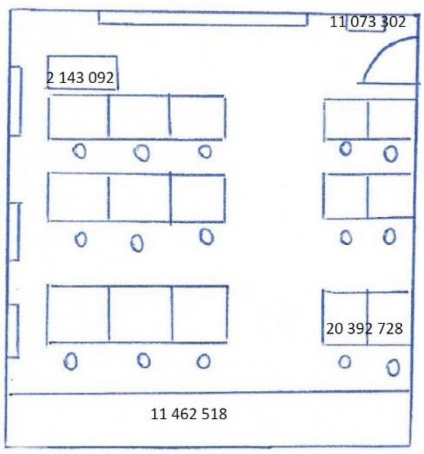
E. Worksheet 3

Exercise 5

On 1 January 2016, Belgium had 11 267 910 inhabitants, 1 187 890 of whom lived in the Brussels Capital Region (abbreviation: BCR). About 0.2% of Belgians came to live in the Brussels Capital Region that year, while about 3.3% left the BCR to live elsewhere in Belgium.

1. Using these data, construct a count and a migration matrix.
2. Assume that the given emigration/immigration persists annually. How many Belgians would then live in the Brussels Capital Region and how many outside it in 2017?
3. And how many in 2030?

Take the sum of the number of Belgians living in the Brussels Capital Region in 2017 and the number of Belgians not living in the Brussels Capital Region in 2030. Find the corresponding number on the map of the classroom and go to the place corresponding to this number and find ...

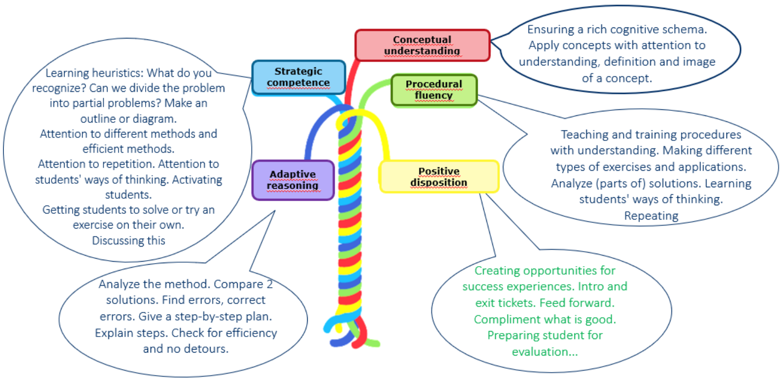


8 Evaluation

In this chapter, we briefly zoom in on in-class evaluation. We start from Kilpatrick's model which outlines the general goals of mathematics education within five intertwining strands of mathematical proficiency. Pursuing these goals requires a different teaching approach and a different way of evaluating. We offer some ideas. After all, evaluation encompasses much more than the purpose of this introductory course.

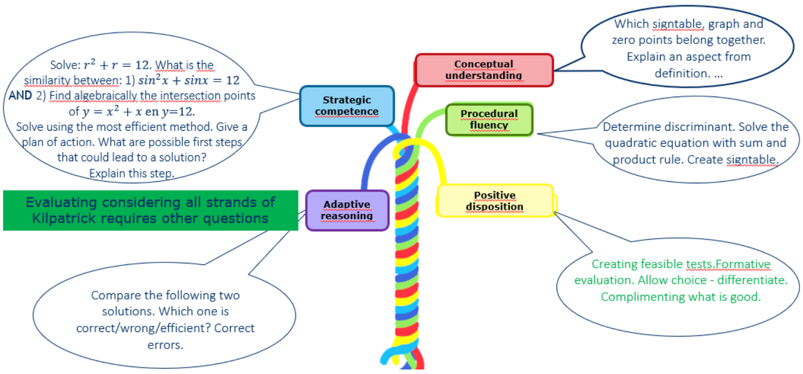
8.1 Teaching approaches in line with evaluation

When designing a lesson, the teacher should think thoroughly about the different lesson objectives. In a lesson series around a particular topic, Kilpatrick's different strands should be adequately addressed. In the 'slide' below some ideas on teaching approaches fit each strand.

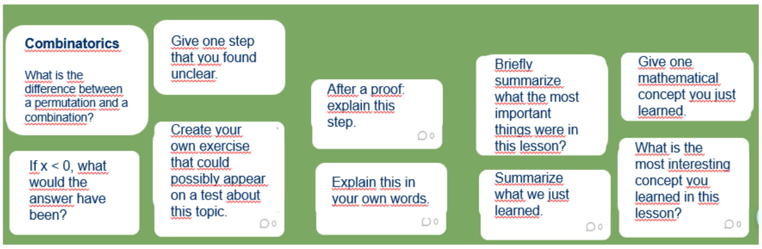


Of course, lesson objectives go hand in hand with evaluating them. The teacher should also focus on this sufficiently during the lesson. Students should be prepared for the evaluation of lesson objectives during the lesson. After all, one of the criteria of quality evaluation is 'transparency'.

The next 'slide' shows some ideas on how to evaluate considering different strands.



Each lesson approach goes in line with evaluation. The teacher should pursue that after each lesson every student knows the new concepts tackled and has insight of them. Ideal the pupil already places new concepts in the right place of the cognitive scheme. Kilpatrick requires e.g.: to train definitions and understanding of new concepts, to train standard procedures, to train research problem solving and modelling, to train explanation of methods… Especially for teachers with few experience the lesson plan is crucial to prepare teaching towards Kilpatricks goals. Activating students is crucial. Asking many questions (involving all pupils) and varying in types of questions is a first important step. It is not an easy skill to ask clear thinking questions that learners can answer, to ask crucial question for steps or concepts where learners might struggle with, to pose two-step model questions and to know how to deal with nonresponse. So, preparing questions at home before teaching the lesson is crucial for a good preparation. In paragraph 1.3 we suggested to divide each lesson in sequences. We add the following suggestion to provide for each lesson sequence a good question which can check whether the pupils understand what was covered in the sequence. A small question that could even be used in a test. 'Did you understand?' or 'Are there any questions?' are questions that are often used but rarely make students think or, more so, rarely prompt students to engage in any activity. They tend to be rather rhetorical questions with no response. In the picture you find some better ideas.



8.2 Coaching pupils

Coaching pupils during the lesson is another important approach to check whether they understand the lesson. The following techniques serve to coach pupils:

- Further Questioning: Ask pupils for an explanation of a certain answer, ask them to explain there method;
- Allow other pupils to evaluate an answer from a pupil;
- Pas questions from pupils on to other pupils;
- Pay attention to build cognitive schemas;
- Ask thinking questions, method questions, crucial questions to make the learning process effective;
- During guidance of group work: avoid lecturing but focus on mini Q&A's, ask questions that move pupils further along in their thinking.

8.3 The role of repetition: intro and exit tickets

'*In the process of teaching math, the repetition of the studied material is given an important place. Properly organized repetition is one of the factors contributing to the intellectual development of each student, their achievement of deep and lasting knowledge. The need to repeat the previously studied material is caused by the very structure of the curriculum of mathematics. For example, students go through the curriculum topic: "Quadrangles" in 8th grade, but use it in grades 10-11 when studying the topics: "Surface of bodies of revolution", "Surface area", "Volume of bodies", etc. To teach mathematics without repeating the material previously passed on every lesson every day, this means transmitting, retelling to students a certain amount of different laws, theorems, formulas, etc., completely not caring about how firmly and consciously this material was mastered by our pets; it means not giving children deep and lasting knowledge. Researches showed us that repetition is the main instrument of the study. Results of the analysis revealed that students exposed to repetition with variation approach had significantly higher achievement, conceptual understanding and improved retention.*' (The role of repetition, Y. Akerke & Y. Ardak, Suleyman D. University, Proceedings of IYSW(2020), vol. 9, p 213-222.)

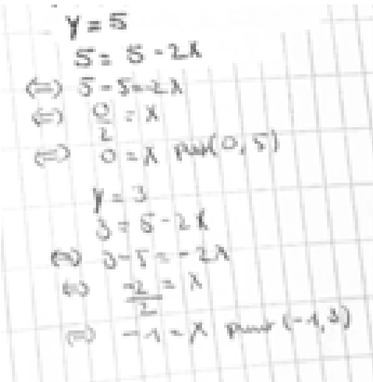
Some ideas to organise a good repetition are:

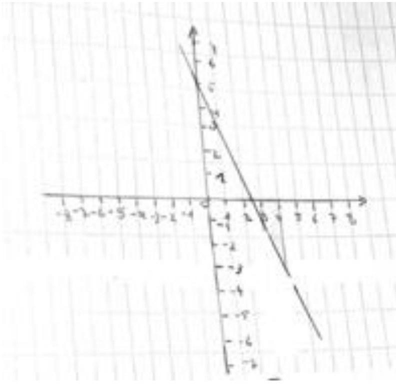
- to brainstorm with pupils about previous knowledge individually or in small groups e.g. via Padlet or big sheets of paper where pupils write down what they still know about a certain topic;
- to use an intro ticket where pupils answer a short question about a previous lesson to start up and orient towards the next lesson;
- to use an exit ticket where pupils answer a short question about the lesson before they exit the classroom.

Some examples to use when developing intro or exit tickets:

- Ask the pupils to write a question about the lesson on one side of a paper and the answer on the other side of the paper. The teacher checks the input of the pupils and can afterwards even use the good ones in any lesson.
- Ask the pupils to pose a question about the content of this lesson, that they might expect on a test.

- Present an exercise an ask the pupils the question ‘Start solving this exercise during three minutes.’ This is just to see whether pupils can make a good start, to check whether they have a good insight in strategies to start solving exercises or questions…
- Give the pupils upon entering the classroom a paper with a question. At the end of the lesson they should be able to answer the question.
- Use pictures of the work of pupils and ask e.g.: ‘Find the mistake in the answer below’ or ‘Describe how to find two points of a straight line in a more efficient way.’

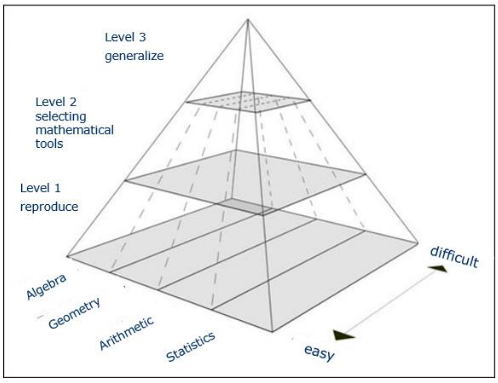




- Ask a question about learning content which was not explicitly covered in the lesson but which you expect to be permanent knowledge.

8.4 The test Pyramid

If we want to develop effective and equitable assessment tools that, moreover, test all aspects of mathematical proficiency (see 3.2), then it is at least important to ask questions about the different strands of mathematical proficiency. A worldwide adopted framework that can undoubtedly help with this is the test pyramid [De Lange, 1999]. The test pyramid has 3 levels. In a balanced test, all three levels are addressed. Although the shape of the pyramid might suggest otherwise, no hierarchy is assumed. The pyramid shape is mainly used to show the distribution of the amount of time spent on tasks at each level. Answering questions at the reproduction level (level 1 - procedural knowledge) usually takes less time than those of questions at level 2 (strategic knowledge) or 3 (conceptual knowledge). At level 1, we find questions with which the learner has had a lot of practice and the answers can be found immediately or in a few steps. In general, for the distribution of questions in a test, the following ratio is used: Level 1: Level 2: Level 3 = 3:2:1



The starting point is that a pupil who has mastered the basic skills but has not shown sufficient understanding of the subject matter covered, will still achieve a passing grade.

We briefly explain the different levels in what follows

Level 1: Reproduce

This level concerns knowledge of concepts, definitions and procedures that have been practised during previous lessons. They are questions corresponding to the strands ‘conceptual understanding’ and ‘procedural fluency’ from Kilpatrick’s model. Pupils should be able to answer these questions easy and quickly.

Sample questions:

- Solve the equation $7x - 3 = 13x + 15$.
- Write $27\ 000\ 000$ in scientific notation.
- What is the domain of the following function: $f(x) = -\sqrt{6-2x} + 4$?

Bear in mind that a task that is a reproduction task for one pupil in a class, because the procedure has been practised a lot, need not be so for a pupil in another class. The teacher can determine which question on a test belongs to which level in the pyramid. Of course, the curriculum objectives for the pupils that should solve the test, give the framework for the teacher to decide upon the correct level.

Level 2: Choosing mathematical tools

This level includes questions on strategic competence and conceptual understanding. Pupils choose the mathematical tools needed to solve a problem. Connections are made between different representations such as being able to use data from a table and a diagram or a graph to answer a question. Pupils can solve simple problems presented within an unfamiliar context.

Some examples:

- Determine the zero points of the function $f(x) = x^2 + 2x - 3$

  Obviously, it is the most efficient to solve this question via the sum and product method, but it can of course also be solved via the ABC formula (discriminant method) or via ICT.

- A plumber must take a straight steel pipe to the ninth floor of a building. Can he take the lift or should he go by stairs?

  The dimensions of pipe and lift are given. The pupil can, for example, draw the situation to scale or use the Pythagorean theorem. The mathematical tools are known, the situation in which they should be used not.

- Kaan wants to make a rectangular box with height 10cm and width 20cm from a piece of cardboard. What dimensions should the cardboard have for the volume to be as large as possible?

  The pupil can draw the situation to get a clear picture of the question. The variable must be chosen which leads towards a function that has to be derived to find a maximum value. If this type of question is regularly made in class it is an easier question than when it is presented for the first time. Usually, pupils have the most difficulty in finding the correct equation for the function.

Level 3: Generalising

Here the questions are about conceptual understanding, strategic competence and adaptive reasoning. More abstract questions belong to this level as well. The pupil needs a insight in all mathematical concepts and strategies and needs to have a positive disposition to maths in order to persevere and get stuck into a problem. Creating and using a mathematical model yourself, reasoning mathematically and prove, reflect, show insight are competences a pupil needs to tackle questions in this level.

When Level 3 questions are not covered in lessons, the teacher cannot expect that pupils can answer these kinds of questions by themselves. It is a skill that the teacher must teach the pupils.

Sample questions include questions where the use of standard procedures is not sufficient and where the pupil has to show understanding of what has been learned.

Some examples:

- Given is the circle with equation $y^2 - 2y + x^2 + 6x - 15 = 0$. We call $M = (a, b)$ the centre of this circle and R the radius. Determine $2a + b + R^2$.

  Source: Ijkingsproef Wiskunde 2014, collaborating Flemish universities.

- Calculate for what value(s) of p the functions $f(x) = 4{,}5x^2 + px$ en $g(x) = -2$ have an intersection point.

From easy to difficult

There is a third dimension in the test pyramid. Questions are classified from easy to difficult. Contrary to what is sometimes thought, a more difficult question does not automatically mean a higher level. A linear equation to be solved can be easy due to integer coefficients but can also contain fractions and therefore be more difficult. Nevertheless, a standard procedure for solving is still used that has been much practised, and the task remains at level 1.

Example: The equation $-3x^2 + 2x + 1 = 0$ is easier to solve than $4x - 5 = 6x^2 - 7$. In the second one the pupil must know the procedure to changes terms so that the left-hand or the right-hand side of the equation becomes 0.

8.5 Formulate questions

A. Avoid multi-layered questions

A very typical phenomenon within mathematics assessment is multi-layered questions. Multi-layered questions are questions that follow one another, where you need outcomes from previous questions to solve a new question. While it can sometimes be useful to test whether pupils can solve a complex question (depending on their level), it is crucial for the validity of a test or exam that the questions are independent as much as possible. Indeed, multi-layered questions have the disadvantage that students get stuck if they cannot solve a particular section of the question or solve it incorrectly.

If you do include a multi-layered question, you can limit the disadvantages of this type of question by providing intermediate results (e.g. the question ‘show that ... equals ...’ instead of ‘calculate ...’), or dividing one question into two independent sub-questions, or by providing a way out if a previous question went wrong (e.g. the message: 'if you didn't find the answer to the previous question, you can ask the teacher, you can then no longer work on that previous question').

If a pupil makes a mistake in a multi layered question most teachers will continue marking. The teacher then checks how the pupil continues to reason and, while marking, calculates along with the error. After all, the student's mathematical reasoning has merit and deserves assessment. Often, however, an error made leads to an unsolvable situation, to a question or solution that is not actually tested and is thus such that the teacher cannot asses it.

Finally: mathematics didactics [Drijvers et al., 2013] also sometimes refers to hidden multi-layered questions or nested questions. These are questions in which students must do many different sub-steps to reach a solution, but which do not look at all like a multi-layered question at first sight because the question itself is not divided into dependent sub-questions. This type of question entails the danger that pupils get stuck quite early in the solution process and thus can no longer show what they can do because they just can't get there. It is important to be aware of this phenomenon and to avoid hidden multi-layered questions as much as possible or at least limit the disadvantages of this question type.

We will now give two examples of typical multi-layered questions and one example of a nested question.

<u>Example 1</u>: multi-layered question

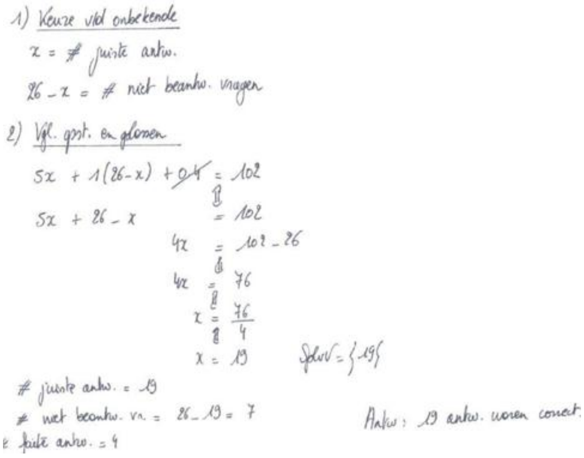
Given are two functions: $f(x) = -x - 4$ and $g(x) = 2x + 1$

1. Determine $g^{-1}(x)$
2. Determine $i(x) = (g \circ f \circ g^{-1})(x)$
3. Determine $i(2)$

The pupil that did not find the answer to sub-question 1, can't solve 2 and 3. It is better not to use g^{-1} in sub-question 2 to evaluate whether the pupil understands the concept of composite functions. In the same way it is better to use one of the given functions to evaluate whether a pupil can calculate a function value in sub-question 3.

Example 2: nested question

The Junior Mathematics Olympiad consists of 30 multiple-choice questions. For each correct answer, you earn 5 points. Of course, you get no points for each incorrect answer, but for each unanswered question, you earn 1 point. Jurgen scored 102 points and gave 4 incorrect answers. How many answers were correct?


1) Keuze vd onbekende

x = # juiste antw.

$26 - x$ = # niet beantw. vragen

2) Vgl. opst. en oplossen

$5x + 1(26-x) + \cancel{0 \cdot 4} = 102$

$\Updownarrow$

$5x + 26 - x = 102$

$4x = 102 - 26$

$\Updownarrow$

$4x = 76$

$\Updownarrow$

$x = \frac{76}{4}$

$\Updownarrow$

$x = 19$

Oplv = {19}

juiste antw. = 19

niet beantw. vr. = 26 − 19 = 7

foute antw. = 4

Antw: 19 antw. waren correct.

In the solution of the teacher you clearly see the three different sub-steps:

1. Model the word problem.
2. Solve the equation.
3. Give the final answer.

The pupil can't continue if the modelling step is not found or in such a way incorrect that there is no equation to be solved. Without a reasonable equation to be solved the answer can't be formulated as well. So this is a poor question to evaluate the 3 different learning objectives.

Example 3: nested question

Solve the inequality algebraically $x^3 - 8x + 8 \geq 0$

Pupils will get stuck when they don't find the zero points. Without the zero points, it is impossible to draw a correct sign table. Without the sign table they can't solve the inequality.

B. Alternatives for multi-layered questions

A general rule is to consider whether it is really necessary to ask a multi-layered question where answers from the previous one are needed to further solve the question. In the **first example** above, we gave equally valid alternatives.

For the **second example** we present a few alternatives.

1. Change the question 'How many answers were correct?' into 'Set up an equation that can solve this problem. You should NOT solve the equation.'
2. Change the question 'How many answers were correct?' into 'Show algebraically that 19 is a correct answer. Formulate the answer forming a good sentence.'
3. Add the following sentence: 'If you can't find the equation, ask your teacher for it.'

In another question the teacher can check whether pupils can solve an equation.

For the **third example** one could change the open question in a closed one (see 6.3). E.g.

For question 1 and 2 is given that $f(x) = x^3 - 8x + 8$

1. Calculate algebraically the zero points of f
2. Make a sign table for f. *If you did not find the zero points in 1 you may use the calculator OR ask your teacher.*

3. Explain why the sign table in 2 can be used to solve $x^3 - 8x + 8 \geq 0$. *If you did not find the sign table, ask your teacher. Afterwards you may not work on sub question 1 and 2.*
4. Solve the inequality $x^3 - 8x + 8 \geq 0$
5. Given the functions $f(x) = x^3 - 4x$ and $g(x) = 4x - 8$, investigat for which values of x the graph of f is below the one of g ($f(x) < g(x)$).

A **fourth alternative** is to add extra questions for pupils that get stuck. E.g.:

Given: $f(x) = -4x^4 - 20x^3 - 13x^2 + 30x - 9$

1. Calculate algebraically and exactly the zero points of the polynomial function.
2. Factorize $f(x)$.

You now solve the next 2 subquestions (3 and 4). However, if you haven't found the zero points, solve question 5 instead of 3 and 4.

3. Make a sign table of the function.
4. Solve: $f(x) < 0$
5. If you haven't found 1 and 2:
   Solve using a sign table: $(2x + 7)^2(-3x + 2)(x^2 + 6x + 5) \leq 0$

C. Ask short questions

Short questions that test whether pupils understand or can solve parts they need to solve a larger or complex question are ideal to use as an intro or exit ticket but also in a test or exam. We give a few examples.

Example 1

Set up the extended matrix of the following system of equations.

$$\begin{cases} 4x - 3y - 5z = 25 \\ x + 5y - 7z = 12 \\ 6x + y - 13z = 43 \end{cases}$$

This is an easy question on solving a system of equations with the Gauss-Jordan method. But, if pupils make already a mistake in de first step both the rest of the solution and the marking work can be hell.

Example 2

Note the solution set of the system of equations with the following' resolved matrix':

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & -11 & 2 \\ 0 & 0 & 0 & 1 & 4 & -7 \end{array}\right]$$

This question tests whether students can formulate the answer to a system with an infinite number of solutions. Formulating a solution set is a difficult step in this kind of systems which students also often forget. By this type of short question one highlights this part of the solution.

Example 3

Make a sketch of what is given and what is asked in the following exercise. You don't have to solve the question: '*Kaan builds wooden bungalows which have a triangular front and back and a rectangular floor. The triangle has a height of 5m and a width of 8m. De rectangular floor has a length of 10m. In the triangular front wall there should be a rectangular glass door as large as possible. What are the dimensions of this glass door*?'

Example 4

Complete:

$$\cos(\pi-\alpha) = \qquad ; \tan(\pi+\alpha) = \qquad ; \sin(-\alpha) =$$

$$\cos\left(\alpha-\frac{\pi}{2}\right) = \qquad ; \ \sin(3\pi-\alpha) = \qquad ; \ \cot\left(\frac{\pi}{2}-\alpha\right) =$$

As a bonus question one could ask: Simplify the following expression:

$$\frac{\cos(\pi-\alpha).\tan(\pi+\alpha).\sin(-\alpha)}{\cos(\alpha-\frac{\pi}{2}).\sin(3\pi-\alpha).\cot(\frac{\pi}{2}-\alpha)} =$$

Sometimes you will find the bonus question on a test or exam. Above this question has been transformed into a closed question. The different parts do not depend on each other and test different objectives. The first 3 sub-questions test the knowledge of the formulas of related angles. The next 3 sub-questions require students to apply these formulas at a basic level. The bonus question tests whether they can enter them properly into a larger whole and whether they can simplify fractions.

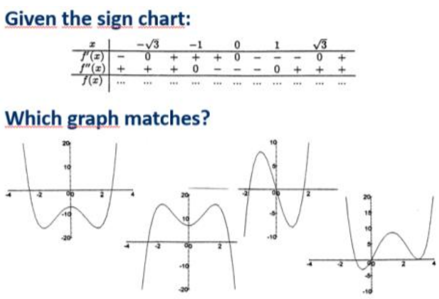
We found this question in an exam of the ‘Examencommissie secundair onderwijs van de Vlaamse Gemeenschap’, an institution that sets exams for everyone who, for one reason or another, has not obtained a diploma in the regular education system, e.g. due to school fatigue or a deliberate choice of home education.

Example 5: multiple choice question

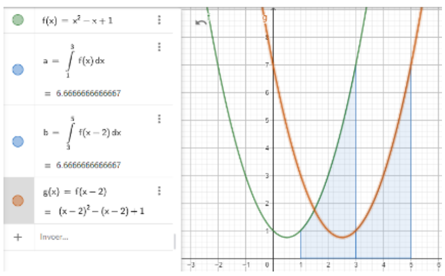
Given $\int_1^3 f(x)dx = 2.4$. What is $\int_3^5 f(x)dx$

1. 0,4
2. 2,4
3. 4.4

Example 6: multiple choice question 2



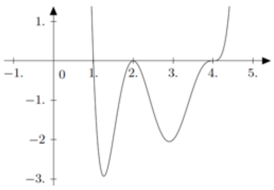
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
$f'(x)$	$-$	0	+	+	+	0	$-$	$-$	$-$	0	+
$f''(x)$	+	+	+	0	$-$	$-$	$-$	0	+	+	+
$f(x)$	...	...	...	...	...	...	...	...	...	...	...



The above multiple-choice questions are good because they test insight. In example 5, one must understand that the second integral can be solved via substitution which entails a translation. Example 6 tests the understanding of the meaning of the first and second derivatives without calculating. Admittedly: the layout of this question could be better. Example 7 below is an example of a special type of multiple-choice. A true-false question gives the pupil two choices. Of course, there is the risk of guessing where the student has a 50% chance of guessing correctly. By asking for an explanation, you increase the quality of the insight question but the question is no longer suitable as a short question.

Example 7: True – False question

Consider the graph. Which operations are true or false



	True	False
a) 2 is a root with even multiplicity.		
b) 4 is a simple root.		
c) 1 is a simple root.		
d) The degree of this polynomial function is at most 3.		
e) The polynomial function can have degree 10.		
f) The polynomial function can have degree 5.		

D. Ask questions that test mathematical insight

Questions that probe conceptual understanding, strategic competence and adaptive reasoning (see 3.2) are usually level 3 questions (see 8.4). In 3.3 we emphasized the importance of pupils building rich cognitive schemas. This is not only a matter of the teaching method in class, but also of assessment practice. It requires a different kind of test questions and a different kind of exercises. We give some examples to inspire.

Examples - pupils explain concepts

Instead of asking definitions that students can literally memorise (which therefore falls more under procedural knowledge and Level 1 questions), the following questions test the understanding of a definition or (part of) an exercise.

Example 1

'The integral is an infinite sum.'

To this statement, we find the following two formulas in the course:

$$\text{if } n \to \infty \text{ then } f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \to \int_a^b f(x)dx$$

Explain the meaning of the term $f(x_2)\Delta x$ of the sum.

Example 2

Each function $f\colon \mathbb{R} \to \mathbb{R} : x \mapsto ax^2 + bx + c$ where $a \in \mathbb{R}_0$ and b, c $\in \mathbb{R}$, is a second degree function.

Why does this definition exclude that $a = 0$?

Example 3

If Elif obtains the solution set $V = \left\{\frac{2\pi}{3} + k \cdot \frac{\pi}{2}\right\}$ for a trigonometric equation and Tigran obtains $V = \left\{\frac{5\pi}{3} - k \cdot \frac{\pi}{2}\right\}$, do they then obtain the same solutions? Explain in your own words.

Examples - pupils assess solutions and justify steps

Instead of asking pupils to solve an exercise, they should assess a given solution. Thus, they learn to justify why certain steps are right/wrong.

Example 1

For which value of x is the following statement true: $5x - 10 = -3x + 30$. Below are the answers of 3 learners. Are the answers right or wrong? Why?

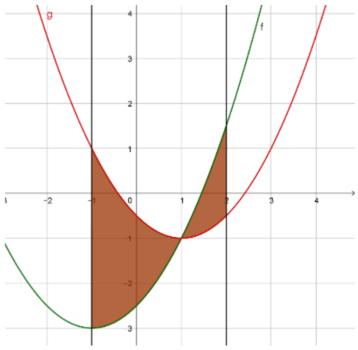
$5x - 10 = -3x + 30$	$5x - 10 = -3x + 30$	$5x - 10 = -3x + 30$
$2x - 10 = 30$	$8x + 10 = 30$	$8x - 10 = 30$
$2x = 20$	$8x = 20$	$8x = 40$
$x = 10$	$x = 2{,}5$	$x = 5$

Source: Handboek Wiskunde didactiek p313 [Drijvers et al., 2013]

Example 2

Evaluate the following method to find the shaded area. Is it right or wrong? Explain your answer and use in your explanation the meaning of the integrals of the given functions.

$$\left[\int_{-1}^{1}\big(g(x) - f(x)\big)dx + \int_{1}^{2}\big(f(x) - g(x)\big)dx\right]$$



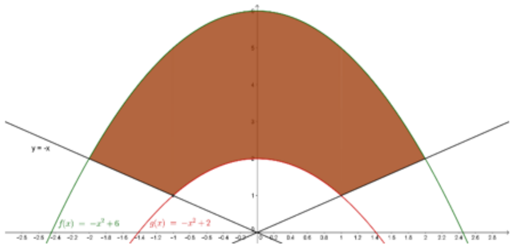
Example - pupils choose or assess efficiency method

Given are the following functions: $y = x$, $y = -x$, $f(x) = -x^2 + 6$ and $g(x) = -x^2 + 2$.

a. Indicate on the graph the points you need to determine the area of the shaded region.

b. Calculate the coordinates the points you need to determine the area of the shaded region.

c. Explain why the following expression with integrals calculate the area of the shaded region.



$$2\left[\int_{0}^{1}\big(-x^2 + 6 - (-x^2 + 2)\big)dx + \int_{1}^{2}(-x^2 + 6 - x)dx\right]$$

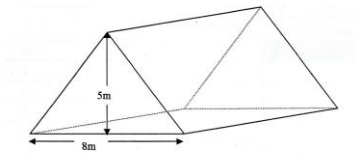
d. Argue whether the method in c. is an efficient one to calculate the shaded area.

E. Ask a step-by-step plan

Instead of asking pupils to solve an exercise, they should give the steps to follow if you have to solve the question.

Example

Make a step-by-step plan to solve the following exercise: '*Kaan builds wooden bungalows of the shape below with height 5m, width 8m and length 10m. In the side wall there will be a rectangular glass door as large as possible. What are the dimensions of this glass door*?'



F. Be economical with questions

When in doubt about the feasibility of a test or examination within the time available, it is best to delete a question or opt for differentiation: students who have answered all the questions and still have some time left can answer an extra question and optionally earn a bonus.

Avoid questions that test the same curriculum objectives. One argument may be to give learners multiple opportunities, but this misses the point: a slightly different question format is unlikely to help a learner who has not achieved a particular (curriculum) goal either. Moreover, this makes tests or exams unnecessarily long and causes time constraints for pupils. The following example gives two questions from the same maths exam for a grade 5 (or grade 11) class in the upper level of a secondary school. We notice that exact the same question is asked twice and, moreover, both questions are nested questions. Although the teacher's intention here was to give students multiple opportunities to show what they can do, the alternative discussed in B is much more suitable.

Question 1

Algebraically determine the zero points and the sign table of the following polynomial function. Write down your complete procedure, only a result does not give you any points.

$$f(x) = 4x^2 + 6x - 1$$

Question 2

Algebraically determine the solution set of the inequality below. Write down your complete procedure, only a result does not give you any points.

$$2x^2 - 2x > x^2 - 3x + 6$$

Avoid double questions. In the example below, the teacher wants to test, on the one hand, whether pupils can solve a simple system of equations using the Gauss-Jordan method. On the other hand, he also wants to test whether they can solve systems with infinitely many solutions. To avoid pupils having to apply the entire Gauss-Jordan method twice to such a system, he therefore only asks about the solution set. This is also the most difficult step in this kind of system.

Question 1

Solve the system belonging to the following matrix:

$$\begin{bmatrix} 5 & 10 & 15 & 50 \\ 15 & 20 & 0 & 50 \\ 10 & 10 & 10 & 50 \end{bmatrix}$$

Question 2

Note the solution set belonging to the following matrix.

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -2 & 5 \\ 0 & 0 & 1 & 0 & -11 & 2 \\ 0 & 0 & 0 & 0 & 4 & -7 \end{bmatrix}$$

G. Question on 'How to start?'

Often when difficult questions are presented in class. The teacher will start up the solution with a Q&A in a classical approach. Once the difficult part of 'starting up' is explored the pupils can continue. In this way they never learn how to start solving and try out methods, reject them, start all over again. So it is better to let pupils try to solve and see what they do. This activity should not take too long in class. Let's say 5- or 10-minutes max. After that a group or classical approach follows whereby different methods are presented, analysed and discussed. What is good? Does the method lead us somewhere? Which previous knowledge is used or is needed?...

The teacher must have the skill to bring together the different methods of the pupils that are good to find a good solution. Where possible, it is very interesting to keep several good solutions. With several good solutions, revisiting which is or are the most efficient reinforces the strand of adaptive reasoning.

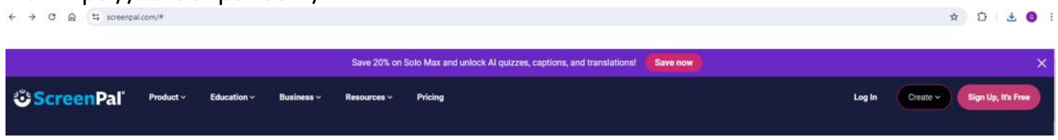
If then on a test pupils also get a question such as 'Give one are to ideas how you can start to solve the following exercise… '. As mentioned above, giving pupils an small test whereby they have to start solving an exercise during three minutes, is another way to highlight the skill to start solving an exercise.

9 ICT – integration

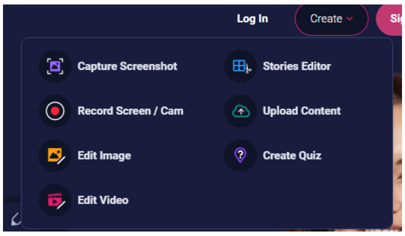
In this paragraph we introduce some useful ICT-tools for maths lessons via step-by step instructions for the reader to grasp the idea of the tool.

A. Screenpal

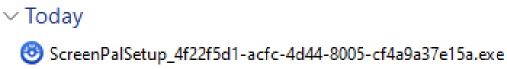
1. Go to https://screenpal.com/



2. Click on 'Create'.



3. Click on 'Record Screen/Cam'. A file starts downloading on your computer.
4. Go to downloads and click on the exe file



5. Screenpal is Starting up.
6. Click on the red REC button and you start recording.

7. You notice it starts counting 3, 2, 1 GO! And you can start recording. Watch this example: https://youtu.be/fcYzKqblT3A
8. Click on the 'pause' button to stop recording. You can listen to what you did.
9. Click on download and browse your computer to choose the map you want to save the file in.
10. Click on 'Publish'. Your video gets a name 'Recording #1' unless if you gave it another name in the previous screen.
11. Click 'Done' and find your video in the map you choose.

B. PhotoMath on smartphone

1. Download the app 'PhotoMath'
2. Scan the following expressions and see what the app does.

$$\frac{d}{dx}(\sqrt{2}\cdot sin(3x)) \qquad \begin{cases} 3x - y = 21 \\ 2x + y = 4 \end{cases}$$

$$\int_1^4 x\,dx \qquad 5x + 10 = 0$$

3. Write the expressions ${}^{2}\log 32$ and $\log_2 32$ on a sheet of paper. Scan both expressions and see what the app does.

C. Desmos

Desmos is a graphical online calculator that is especially popular in America. The state of Texas itself has allowed it as an alternative to the graphical calculator. There is also an app for the smartphone.

1. Go to www.desmos.com
2. Choose some exercise to do:
   a. Plot the following graph
   $y = 3x^2 - 2x$
   b. Plot the following system of equations:
   $\begin{cases} y = x + 1 \\ x^2 + y^2 = 9 \end{cases}$
   c. Calculate the following integral:
   $\int_2^5 \sin(x)\,dx$

D. Socrative

With Socrative, you can create a quiz and organize different activities from the quiz. Pupils participate in the activity on a computer, tablet or smartphone and do not need to register. They log into the teacher's room. The teacher can track how pupils answer during or after practice time. You can also set the quiz as homework.

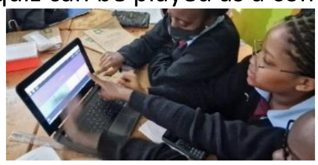
1. Go to www.socrative.com
2. Create an account and log in.
3. Go to 'Library' and create a quiz.
4. Launch a space race.
5. If you still have time:
   a. Test out your space race among yourselves.
   b. Try other possibilities on the same quiz.

E. Quizizz

With Quizizz, you can organize a quiz where pupils compete against each other in solving pre-set questions as quickly as possible. Pupils play on a computer, tablet or smartphone and do not have to register to join the quiz. The teacher receives a summary of pupils' answers after a quiz. You can also give the quiz as homework.

1. Watch 10” of the following video (1'17 - 1'27): (23) PR Internationalisering 2223 DEF - YouTube
2. Create an account on Quizizz at www.quizizz.com
3. Create a quiz with different types of questions. Incorporate at least 3 of the following types:
   a. 1 multiple choice question,
   b. 1 checkbox question,
   c. 1 fill-in-the-blank,
   d. 1 open-ended question,

   Need help? Watch the following video https://www.youtube.com/watch?v=52Y0FUNjNBs
   And/or https://www.youtube.com/watch?v=_Ct8X6VrEuE
4. Still have time? Make sure the quiz can be played as a competition between pupils.


F. Padlet

A 'Padlet' is an association board that can be used in your lessons.

1. Go to https://nl.padlet.com and register.
2. Create a Padlet for a question that you could use in a lesson.

G. QR-codes

QR codes stand for quick response. Just like barcodes in the supermarket, you scan in QR codes. For example, a QR code can contain a link to a website, which you then visit immediately when you scan the QR code. Educationally, this can be very useful to make your written course more accessible with QR codes (making solution keys available, working differentiated...).

1. Look up a "QR generator" via the Internet.
2. Create a QR code for e.g. an educational YouTube video, an interesting website, a message you make....
3. Download the file with the QR code image and place it in a Word document.
4. Download a QR reader on your cell phone and try out your code.

H. ChatGPT

1. Go to chat.openai.com and "register" or "sign in"/"log in" if you have an account.
2. Ask the following question: 'Is the derivative of the function $f(x) = Arcsinx$, equal to $f'(x) = 3x^2$?'
3. Ask a mathematical question. Analyze the answer critically and ask a side question if necessary think about a good prompt.
4. Insert the following prompt: 'Give me incorrectly solved exercises on second grade equations at the level of a 15-year-old Zambian/SouthAfrican girl.' Respond that you disagree with the answer and say why. Analyze ChatGPT's response.