The multiplication of matrices

Text based on 'Matrix applications' (in dutch)

From: ‘Wiskunde vanuit toepassingen: functies en matrices als modellen’, J. Roels, J. Deprez, D. Janssens, D. De Bock; 1990 KU Leuven Aggregatie hoger secundair onderwijs wiskunde. p164-168)

The multiplication of matrices is not so obvious. Besides, why are matrices multiplied in this way? When we ask students to find a meaningful definition for the multiplication of matrices, no one will suggest the "right" method. Also, can't students discover this multiplication on their own? Some Flemish maths textbooks apply multiplication with stock matrices. In the following worktext, students discover for themselves the method for multiplying matrices from a market study context. Before proceeding to the general multiplication after task 12, it is advisable to discuss some concrete examples so that students get the feeling that this method of multiplication is common.

Worksheet: A market study

A consumer association runs the following promotion. For each new member, it determines which department store is the cheapest for weekly purchases (a study has shown that it is not advantageous to buy the cheapest products in each department store: the journey to the different department stores must be included in the total expenditure). For this, the consumer association tallied the prices of a large number of basic products in different department stores. This can be found in the following table:

Products	**Dellijs**	**Soussi Market**	**Heebee**	**Ilda-market**
Sweet potatoes (€/kg)	3,5	2,5	4	3
Carrots (€/kg)	1,59	1,4	1,3	1,24
Bread (€/piece)	2,55	1,85	2,29	2,9
Sparkling water (€/bottle 1l)	1	0,85	0,95	0,9
Feta cheese (€/kg)	16,09	15,77	25	16,99

1. How much do you pay for 5 kg of sweet potatoes in Soussi Market?

In practice, of course, it involves many more products and more department stores. The data will therefore be processed by computer. We will look at this limited part so that we can recalculate the calculations by hand. When registering as a new member of the consumer association, you can tell us how much you use of these basic products per week. One then calculates which department store is the cheapest for these purchases. Suppose, for example, that the Dusangabe family uses the following quantities of these products on average per week:

- 3kg sweet potatoes
- 2kg carrots
- 10 loaves of bread
- 5 bottles of sparkling water
- 0.5kg of feta cheese

2. Choose a department store by circling it in this list: Dellijs, Soussi Market, Heebee, Ilda market.

3. Calculate how much the Dusangabe family, in terms of these products, would spend per week in the department store of your choice.

4. Look up classmates who chose a different department store to narrow down your arithmetic and solve the following question: Based on your calculations, which department store would you recommend family Dusangabe and why?

Matrix notation

To solve question 4, you had to do 4 calculations. In practice, you obviously must do many more. With a computer, however, this does not cause any problems. We will look for a systematic way of calculating so that this can be automated and thus programmed.

5. The data with the different prices of products can be kept clearly in a matrix. Create such a matrix. You have a lot of choices to do this.

Check whether you recognise your matrix in the following matrix that we will continue working with in this worktext:

$$\begin{bmatrix} 3{,}5 & 1{,}59 & 2{,}55 & 1 & 16{,}09 \\ 2{,}5 & 1{,}4 & 2{,}29 & 0{,}95 & 15{,}77 \\ 4 & 1{,}3 & 1{,}85 & 0{,}85 & 25 \\ 3 & 1{,}24 & 2{,}9 & 0{,}9 & 16{,}99 \end{bmatrix}$$

6. In each case, fill in below which information is in the corresponding row or column.

   a. Rows:
      Row 1: Row 2:
      Row 3: Row 4:

   b. Columns:
      Column 1: Column 2:
      Column 3: Column 4: Column 5:

7. You can also record "consumption" and "expenditure" in a matrix. Make some suggestions.

   a. Consumption

   b. Expenditure

8. In 7, you used a special matrix. Circle the ones you made:

   a. Consumption: column matrix, row matrix

   b. Expenditure: column matrix, row matrix

In what follows, we continue with the following matrices:

consumption expenditure

$$\begin{bmatrix} 3 \\ 2 \\ 10 \\ 5 \\ 0.5 \end{bmatrix} \qquad \begin{bmatrix} 52{,}23 \\ 45{,}84 \\ 49{,}85 \\ 53{,}48 \end{bmatrix}$$

We find the expenditure matrix by combining the price matrix with the consumption matrix of family Dusangabe: this is called multiplying.

$$\begin{array}{r} \\ \text{Dellijs} \\ \text{Soussie} \\ \text{Heedee} \\ \text{Ilda} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{sweet p} & \text{carrot} & \text{bread} & \text{s water} & \text{feta cheese} \end{array} \\ \begin{pmatrix} 3.5 & 1.59 & 2.55 & 1 & 16.09 \\ 2.5 & 1.4 & 2.29 & 0.95 & 15.77 \\ 4 & 1.3 & 1.85 & 0.85 & 25 \\ 3 & 1.24 & 2.9 & 0.9 & 16.99 \end{pmatrix} \end{array} \cdot \begin{pmatrix} 3 \\ 2 \\ 10 \\ 5 \\ 0.5 \end{pmatrix} \begin{array}{l} \text{kg sweet potatoes} \\ \text{kg carrots} \\ \text{loaf of bread} \\ \text{bottles sparkling water} \\ \text{kg feta cheese} \end{array}$$

$$= \begin{bmatrix} 52{,}23 \\ 45{,}84 \\ 49{,}85 \\ 53{,}48 \end{bmatrix} \begin{array}{l} \text{Dellijs} \\ \text{Soussie} \\ \text{Heedee} \\ \text{Ilda} \end{array}$$

Of course, one will not always mention products and department stores.

How do we find, for example, the third element of the expenditure matrix, the expenditure in Heedee? To do this, we need to multiply the elements of the third row of the price matrix (the prices in Heedee) by the corresponding elements from the consumption matrix and add the results.

Thus, the total expenditure in Heedee is: $4 \cdot 3 + 1{,}3 \cdot 2 + 1{,}85 \cdot 10 + 0{,}85 \cdot 5 + 25 \cdot 0{,}5 = 49.85$

The calculation of the full expenditure matrix of the Dusangabe family (with matrixcalc.org):

$$\begin{pmatrix} 3.5 & 1.59 & 2.55 & 1 & 16.09 \\ 2.5 & 1.4 & 2.29 & 0.95 & 15.77 \\ 4 & 1.3 & 1.85 & 0.85 & 25 \\ 3 & 1.24 & 2.9 & 0.9 & 16.99 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 10 \\ 5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 3.5\cdot 3+1.59\cdot 2+2.55\cdot 10+1\cdot 5+16.09\cdot (0.5) \\ 2.5\cdot 3+1.4\cdot 2+2.29\cdot 10+0.95\cdot 5+15.77\cdot (0.5) \\ 4\cdot 3+1.3\cdot 2+1.85\cdot 10+0.85\cdot 5+25\cdot (0.5) \\ 3\cdot 3+1.24\cdot 2+2.9\cdot 10+0.9\cdot 5+16.99\cdot (0.5) \end{pmatrix} = \begin{pmatrix} 52.23 \\ 45.84 \\ 49.85 \\ 53.48 \end{pmatrix}$$

We take in a second family. The consumption matrix of the Scheldeman family is:

$$\begin{bmatrix} 2{,}5 \\ 1 \\ 8 \\ 0 \\ 0.6 \end{bmatrix}$$

9. Calculate the expenditure matrix.

10. Which department store is the most advantageous for the Scheldeman family?

11. Given are the following matrices A and B, calculate $A.B$

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

12. Given is the matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

a. How many elements must a column matrix B have in order to be multiplied by A?

b. Which dimension has $A \cdot B$ then?

c. What is the dimension of the product if you multiply a matrix with dimension 1×8 by a matrix with dimension 8×1?

The multiplication of matrices

We return to the consumer association's promotional action. The association also wants to use these data and results itself for a general study. Therefore, they will "keep" everything in matrices.

From the consumption of the Dusangabe and Scheldeman families, for example, we can make one matrix.

$$\begin{array}{cl} \begin{bmatrix} 3 & 2{,}5 \\ 2 & 1 \\ 10 & 8 \\ 5 & 0 \\ 0{,}5 & 0.6 \end{bmatrix} & \begin{array}{l} \text{kg sweet potatoes} \\ \text{kg carrots} \\ \text{loaf of bread} \\ \text{bottles sparkling water} \\ \text{kg feta cheese} \end{array} \\ \text{Dusangabe} \quad \text{Scheldeman} & \end{array}$$

We can also combine the expenditure columns into one matrix.

$$\begin{array}{cl} \begin{bmatrix} 52{,}23 & 40{,}39 \\ 45{,}84 & 35{,}43 \\ 49{,}85 & 41{,}1 \\ 53{,}48 & 42{,}13 \end{bmatrix} & \begin{array}{l} \text{Dellijs} \\ \text{Soussie Market} \\ \text{Heedee} \\ \text{Ilda} \end{array} \\ \text{Dusangabe} \quad \text{Scheldeman} & \end{array}$$

Both calculations of multiplying a matrix and a column thus give one product calculation of two matrices.

$$\begin{bmatrix} 3{,}5 & 1{,}59 & 2{,}55 & 1 & 16{,}09 \\ 2{,}5 & 1{,}4 & 2{,}29 & 0{,}95 & 15{,}77 \\ 4 & 1{,}3 & 1{,}85 & 0{,}85 & 25 \\ 3 & 1{,}24 & 2{,}9 & 0{,}9 & 16{,}99 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2{,}5 \\ 2 & 1 \\ 10 & 8 \\ 5 & 0 \\ 0{,}5 & 0.6 \end{bmatrix} = \begin{bmatrix} 52{,}23 & 40{,}39 \\ 45{,}84 & 35{,}43 \\ 49{,}85 & 41{,}1 \\ 53{,}48 & 42{,}13 \end{bmatrix}$$

13. Take the number on the third row and in the second column of the expenditure matrix.

a. What is the significance of this number in the context of this market study?

b. How can you calculate it?

c. If you didn't in b. articulate the way to calculate this number in matrix notation. So use the names rows, columns, elements ...
A tip: you may still need the following concretisation. Write down the names of shops, products and families at the third row and second column.

14. Calculate in this way $A \cdot B$

$$A = \begin{bmatrix} 2 & 4 & 0 & 1 \\ 1 & 2 & 5 & -1 \\ 2 & 3 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 5 \\ 2 & 1 \\ 3 & 0 \\ -3 & 3 \end{bmatrix}$$

15. Given is the matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{bmatrix}$$

   a. What dimension must a matrix B have to compute $A \cdot B$? Then what is the dimension of $A \cdot B$?
   b. What dimension must a matrix B have to compute $B \cdot A$? Then what is the dimension of $B \cdot A$?
   c. Under what dimension conditions can you multiply matrices in this way?

16. We return to the promo one last time. To the matrix you set up in question 5 (take this A), give a consumption matrix B so that the product is$A \cdot B$ of $B \cdot A$ gives a matrix with the expenditures of the families Dusangabe and Scheldeman.

Solutions

With question 9

$$\begin{pmatrix} 3.5 & 1.59 & 2.55 & 1 & 16.09 \\ 2.5 & 1.4 & 2.29 & 0.95 & 15.77 \\ 4 & 1.3 & 1.85 & 0.85 & 25 \\ 3 & 1.24 & 2.9 & 0.9 & 16.99 \end{pmatrix} \cdot \begin{pmatrix} 2.5 \\ 1 \\ 8 \\ 0 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 3.5\cdot(2.5)+1.59\cdot 1+2.55\cdot 8+1\cdot 0+16.09\cdot(0.6) \\ 2.5\cdot(2.5)+1.4\cdot 1+2.29\cdot 8+0.95\cdot 0+15.77\cdot(0.6) \\ 4\cdot(2.5)+1.3\cdot 1+1.85\cdot 8+0.85\cdot 0+25\cdot(0.6) \\ 3\cdot(2.5)+1.24\cdot 1+2.9\cdot 8+0.9\cdot 0+16.99\cdot(0.6) \end{pmatrix} = \begin{pmatrix} 40.39 \\ 35.43 \\ 41.1 \\ 42.13 \end{pmatrix}$$

In the calculation above question 13

$$\begin{pmatrix} 3.5 & 1.59 & 2.55 & 1 & 16.09 \\ 2.5 & 1.4 & 2.29 & 0.95 & 15.77 \\ 4 & 1.3 & 1.85 & 0.85 & 25 \\ 3 & 1.24 & 2.9 & 0.9 & 16.99 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2.5 \\ 2 & 1 \\ 10 & 8 \\ 5 & 0 \\ 0.5 & 0.6 \end{pmatrix} = \begin{pmatrix} 3.5\cdot 3+1.59\cdot 2+2.55\cdot 10+1\cdot 5+16.09\cdot(0.5) & 3.5\cdot(2.5)+1.59\cdot 1+2.55\cdot 8+1\cdot 0+16.09\cdot(0.6) \\ 2.5\cdot 3+1.4\cdot 2+2.29\cdot 10+0.95\cdot 5+15.77\cdot(0.5) & 2.5\cdot(2.5)+1.4\cdot 1+2.29\cdot 8+0.95\cdot 0+15.77\cdot(0.6) \\ 4\cdot 3+1.3\cdot 2+1.85\cdot 10+0.85\cdot 5+25\cdot(0.5) & 4\cdot(2.5)+1.3\cdot 1+1.85\cdot 8+0.85\cdot 0+25\cdot(0.6) \\ 3\cdot 3+1.24\cdot 2+2.9\cdot 10+0.9\cdot 5+16.99\cdot(0.5) & 3\cdot(2.5)+1.24\cdot 1+2.9\cdot 8+0.9\cdot 0+16.99\cdot(0.6) \end{pmatrix} = \begin{pmatrix} 52.23 & 40.39 \\ 45.84 & 35.43 \\ 49.85 & 41.1 \\ 53.48 & 42.13 \end{pmatrix}$$

With question 16

$$\begin{pmatrix} 3 & 2 & 10 & 5 & 0.5 \\ 2.5 & 1 & 8 & 0 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 3.5 & 2.5 & 4 & 3 \\ 1.59 & 1.4 & 1.3 & 1.24 \\ 2.55 & 2.29 & 1.85 & 2.9 \\ 1 & 0.95 & 0.85 & 0.9 \\ 16.29 & 15.77 & 25 & 16.99 \end{pmatrix} = \begin{pmatrix} 52.33 & 45.84 & 49.85 & 53.48 \\ 40.51 & 35.43 & 41.1 & 42.13 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 10 & 5 & 0.5 \\ 2.5 & 1 & 8 & 0 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 3.5 & 2.5 & 4 & 3 \\ 1.59 & 1.4 & 1.3 & 1.24 \\ 2.55 & 2.29 & 1.85 & 2.9 \\ 1 & 0.95 & 0.85 & 0.9 \\ 16.29 & 15.77 & 25 & 16.99 \end{pmatrix} = \begin{pmatrix} 3\cdot(3.5)+2\cdot(1.59)+10\cdot(2.55)+5\cdot 1+0.5\cdot(16.29) & 3\cdot(2.5)+2\cdot(1.4)+10\cdot(2.29)+5\cdot(0.95)+0.5\cdot(15.77) & 3\cdot 4+2\cdot(1.3)+10\cdot(1.85)+5\cdot(0.85)+0.5\cdot 25 & 3\cdot 3+2\cdot(1.24)+10\cdot(2.9)+5\cdot(0.9)+0.5\cdot(16.99) \\ 2.5\cdot(3.5)+1\cdot(1.59)+8\cdot(2.55)+0\cdot 1+0.6\cdot(16.29) & 2.5\cdot(2.5)+1\cdot(1.4)+8\cdot(2.29)+0\cdot(0.95)+0.6\cdot(15.77) & 2.5\cdot 4+1\cdot(1.3)+8\cdot(1.85)+0\cdot(0.85)+0.6\cdot 25 & 2.5\cdot 3+1\cdot(1.24)+8\cdot(2.9)+0\cdot(0.9)+0.6\cdot(16.99) \end{pmatrix}$$

$$\begin{pmatrix} 2.5 & 1 & 8 & 0 & 0.6 \\ 3 & 2 & 10 & 5 & 0.5 \end{pmatrix} \cdot \begin{pmatrix} 3.5 & 2.5 & 4 & 3 \\ 1.59 & 1.4 & 1.3 & 1.24 \\ 2.55 & 2.29 & 1.85 & 2.9 \\ 1 & 0.95 & 0.85 & 0.9 \\ 16.29 & 15.77 & 25 & 16.99 \end{pmatrix} = \begin{pmatrix} 40.51 & 35.43 & 41.1 & 42.13 \\ 52.33 & 45.84 & 49.85 & 53.48 \end{pmatrix}$$

$$\begin{pmatrix} 2.5 & 1.4 & 2.29 & 0.95 & 15.77 \\ 3 & 1.24 & 2.9 & 0.9 & 16.99 \\ 3.5 & 1.59 & 2.55 & 1 & 16.29 \\ 4 & 1.3 & 1.85 & 0.85 & 25 \end{pmatrix} \cdot \begin{pmatrix} 2.5 & 3 \\ 1 & 2 \\ 8 & 10 \\ 0 & 5 \\ 0.6 & 0.5 \end{pmatrix} = \begin{pmatrix} 35.43 & 45.84 \\ 42.13 & 53.48 \\ 40.51 & 52.33 \\ 41.1 & 49.85 \end{pmatrix}$$