The Pythagorean theorem
Chapter 2 of the book 'From question to question'.

You may have come across a famous property of right-angled triangles known as the Pythagorean theorem. We begin this chapter with the rediscovery of this theorem and why it was so popular with the builders of ancient Babylon and Egypt. We will then follow in the footsteps of Pythagoras and his disciples to understand the reasons for the emergence of new numbers: the irrationals. We will also offer a little calculation on these new numbers and finish with an important application of the theorem.

Pythagoras was born on Samos, a small island near Asia Minor, in the 6th century BC. Around 530 BC, he settled in Crotone, in southern Italy. There he founded a community, known as the Pythagoreans, which was both religious and political, organised on an egalitarian model. It promoted moral and civic virtues such as courage, austerity, self-control, moderation, effort and collective discipline.

The Pythagoreans also attributed a mystical role to numbers in the context of religion. They believed that the universe and everything in it could be explained using natural numbers. Conceived in this way, mathematics went far beyond the practical recipes used by craftsmen, merchants and navigators.

The ancient Greek cities cultivated the practice of public debate, which was often passionate. They discussed major issues of general interest. For this reason, the art of persuasion was perfected. It is likely that it was because they were imbued with this culture of argumentation that Pythagoras and the Greeks turned mathematics into a demonstrative science, i.e. a science in which you must convince others of the accuracy of your assertions.

Towards the statement of the theorem

Problem 1
Consider an isosceles right-angled triangle and the squares on its sides. Can the two small squares be cut into pieces and the large square be put back together using these pieces?

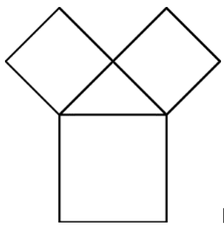

Figure 1

Cut each of the two small squares along a diagonal. The four overlapping triangles thus obtained fill, without overlapping, the large square (Figure 2). So, the area of the large square built on the hypotenuse of the isosceles triangle is the sum of the areas of the squares built on the other two sides.

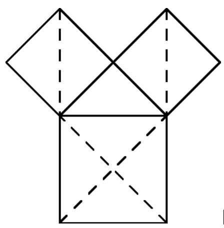

Figure 2

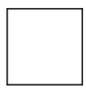

Figure 3

Problem 2
How do you construct a square whose area is twice the area of the original square?

In problem 1, the two squares constructed on equal sides of the isosceles triangle are identical, and the area of the large square is double the area of each of the small squares. This suggests one or other of the following constructions to solve this problem (Figure 4).

Figure 4

Problem 3
We saw in the solution to Problem 1 that the area of the square constructed on the hypotenuse (which is the longest side) of an isosceles right-angled triangle is equal to the sum of the areas of the squares constructed on the other two sides. Do other triangles also have this property?

Let's look first at a few cases of isosceles triangles (Figure 5).

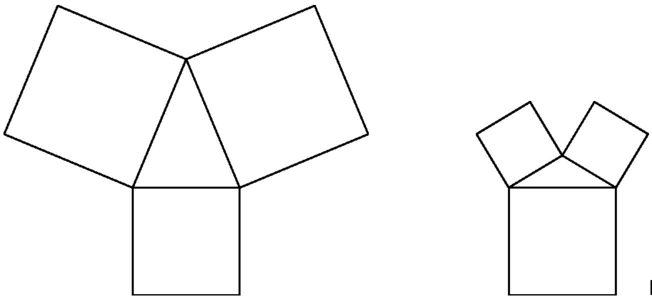

Figure 5

The sum of the areas of the squares on the short sides is either smaller or larger than the area of the square on the third side.

Next, consider non-isosceles triangles. Take, for example, any triangle and a right-angled triangle (Figure 6).

Figure 6

In this case, it seems, by measurement and calculation, that only the right-angled triangle has any chance of possessing the property in question. But how can this be verified in general, and in particular when the dimensions of the triangle are much larger than a sheet of paper in a book or notebook?

At this stage of our research, the property mentioned above is still only a conjecture, i.e. a property that we think is true but of which we are not entirely sure. Let's look for proofs of the conjecture; when we have found one, it will become a theorem.

The next question briefly returns to the isosceles triangle of Problem 1 to prepare arguments for the proof for all right-angled triangles.

Problem 4
Draw twice the same square whose side length is equal to the sum of the side lengths of the right angle of the given isosceles right-angled triangle. Arrange four triangles identical to the isosceles right triangle so that one shows the two squares constructed on the sides of the right angle, and the other shows the square constructed on the hypotenuse.

This construction provides further proof that, in the case of isosceles right triangles, the sum of the areas of the squares constructed on the sides of the right angle is equal to the area of the square constructed on the hypotenuse (Figure 7): in fact, the coloured areas are equal.

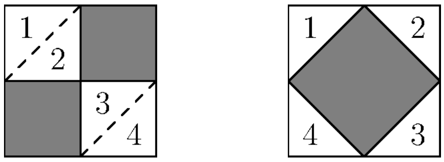

Figure 7

Can you imagine two such arrangements if the right-angled triangle is not isosceles?

Problem 5
Cut out four copies of a right-angled triangle of your choice and draw twice a square whose side length is equal to the sum of the lengths of the sides of the right angle of your triangle.
Is it also possible to arrange the four triangles in these squares in a way that proves the Pythagorean property?

Using Figure 7 as a starting point, we obtain Figure 8. In the square on the left, the coloured space, complementary to that occupied by the four triangles, is equal to the space occupied by the squares built on the sides of the right angle of the initial right-angled triangle. In the square on the right, the coloured space not occupied by the same four triangles is equal to the square built on the hypotenuse.

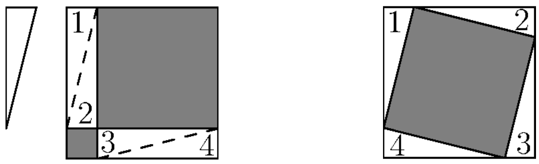

Figure 8

The construction in Figure 8 can be reproduced for any right-angled triangle. The result is therefore established for all right-angled triangles and is known as the Pythagorean Theorem:

In a right-angled triangle, the area of the square constructed on the hypotenuse is equal to the sum of the areas of the squares constructed on the other two sides.

The Pythagorean theorem in terms of lengths

The version of the Pythagorean theorem given in the solution to problem 1 is the most useful in applications.

Problem 1
The area of one square is the sum of the areas of two other squares. These two squares have sides of 30 cm and 40 cm. What is the side of the first square?

Add the areas of the two squares with given sides to obtain the area of the square with unknown side:

$$30^2 + 40^2 = 900 + 1\,600 = 2\,500 = 50^2.$$

Since the area of this square is 2500 cm², this means that its side measures 50 cm.

This question suggests another formulation of the Pythagorean theorem.

Let a and b be the lengths of the two sides of a right-angled triangle and c the length of its hypotenuse. Then the numbers a, b and c satisfy the following equality:

$$a^2 + b^2 = c^2.$$

In the next question, we will discover a method that enabled the builders of Babylon and ancient Egypt to construct walls at right angles.

Problem 2
To check the right angle between two walls, a bricklayer proceeds as follows: starting from a point C on the intersecting edge of the two walls, he marks a point A situated horizontally 30 cm from C (on one of the two walls) and similarly, on the other wall, a point B 40 cm from C. He then measures the length of the wire stretched between the two marks (Figure 9).
What should the result of his measurement be if the angle is indeed right-angled? Why or why not?

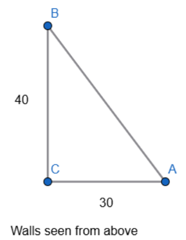

Figure 9

If the angle between the two walls is right-angled, then triangle ABC is right-angled at C and the hypotenuse $[AB]$ measure 50 cm. You can only build one triangle whose sides measure 30, 40 and 50 cm. (Math 1, VI 3, p.254). We can see for ourselves by building it to scale, using the compass.

In the problem of the mason's square, we established the following result: if in a triangle the lengths of the sides satisfy the relation $30^2 + 40^2 = 50^2$, then this triangle is a right-angled triangle whose hypotenuse measures 50 cm. We can generalise this result:

If the distances between three points A, B and C are equal to: $|CA|^2 + |CB|^2 = |AB|^2$, then triangle ABC is right-angled at C.